# Tools for modeling equilibria

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# Variational Inequality

Observe the following containers with force fields. What would be the stationary (equilibrium) position of a ball left at a random location in the container?

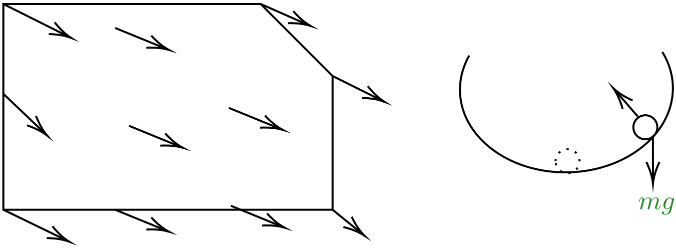



Figure: Containers with force field

You will notice that if the object is at the boundary of the container, it will be unmoved if and only if the force field makes an obtuse or right angle with all the boundary directions.

# Variational Inequality

Definition (Variational Inequality Problem (VIP)). Let $X \subseteq \mathbb{R}^n$ be a nonempty, compact, and convex set and $F : X \mapsto \mathbb{R}^n$. The variational inequality problem $\mathsf{VI}(X, F)$ is to find a vector $\mathbf{x}^*$ such that

$$\boxed{F(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X} \quad (1)$$

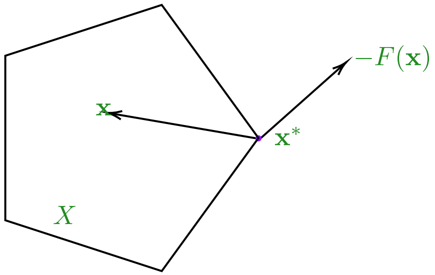



Figure: Geometric interpretation

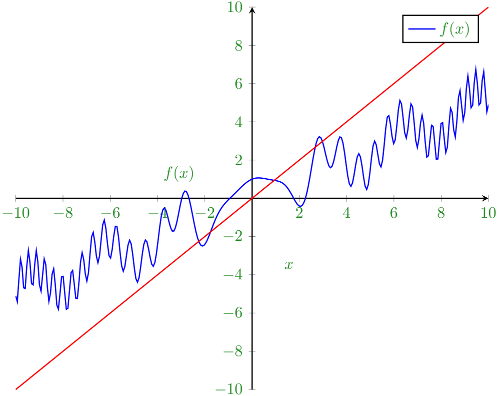
Let $\Omega$ be the set of solutions of $\mathsf{VI}(X, F)$.

Remark. Here, the function $F$ is taken as the negative of the force field.

# Fixed Point Problem

Definition (Fixed Point Problem (FPP)). Let $X \subset \mathbb{R}^n$, The fixed point problem is to find an $\mathbf{x}^* \in X$ such that

$$\boxed{F(\mathbf{x}^*) = \mathbf{x}^*} \qquad (2)$$



## Theorem (Brouwer's Fixed Point Theorem)

*Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact ,and convex set and $F : X \mapsto X$ be a continuous function. Then, there exists at least one point $\mathbf{x}^* \in X$ such that $F(\mathbf{x}^*) = \mathbf{x}^*$.*

The generalization of above theorem for set valued maps is given below:

## Theorem (Kakutani's Fixed Point Theorem)

*Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact ,and convex set and $f : X \mapsto 2^X$ be a set-valued map with the following properties:*

- $F(\mathbf{x})$ *has closed graph i.e.,* $\{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in X$ *and* $\mathbf{y} \in F(\mathbf{x})\}$ *is closed.*
- $F(\mathbf{x})$ *is non-empty for each* $\mathbf{x} \in X$
- $F(\mathbf{x})$ *is convex* $\mathbf{x} \in X$

*Then, there is at least one fixed point, i.e.,* $\exists \mathbf{x} \in X$ *such that* $\mathbf{x} \in F(\mathbf{x})$

Remark. Although these theorems prove existence of a fixed point, they do not give any clue on how to find a fixed point. On the other hand, Banach's Fixed Point Theorem (Contraction Mapping Theorem) defined for contraction mappings gives an iterative method to find the unique fixed point.

# VI and FPP

## Theorem

*Assume that* $X \subseteq \mathbb{R}^n$ *is a nonempty, closed, and convex set and* $F : X \mapsto \mathbb{R}^n$*. Then,* $\boxed{\mathbf{x}^* \in \Omega \iff \mathbf{x}^* = \textbf{proj}_X(\mathbf{x}^* - F(\mathbf{x}^*))}$

## Proof.

Assume that $\mathbf{x}^* \in \Omega$, i.e.,

$$-F(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \leq 0, \forall \mathbf{x} \in X \tag{3}$$

Adding $\mathbf{x}^{*T}(\mathbf{x} - \mathbf{x}^*)$ to both sides, we obtain

$$(\mathbf{x}^* - F(\mathbf{x}^*))^T(\mathbf{x} - \mathbf{x}^*) \leq \mathbf{x}^{*T}(\mathbf{x} - \mathbf{x}^*) \tag{4}$$

$$\implies [\mathbf{x}^* - (\mathbf{x}^* - F(\mathbf{x}^*))]^T (\mathbf{x} - \mathbf{x}^*)) \leq 0 \tag{5}$$

$$\implies \mathbf{x}^* = \textbf{proj}_X(\mathbf{x}^* - F(\mathbf{x}^*)) \tag{6}$$

using theorem proved in previous deck of slides (see theorem on projections). Conversely, the reverse argument can be used. □

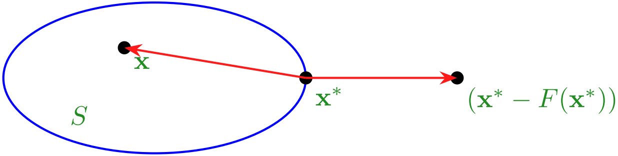



Figure: Geometric interpretation of previous theorem

# Existence of solution of **VI**$(X, F)$

## Proposition

*If $X$ is a nonempty, closed, and convex set and $F : X \mapsto X$ is a continuous function on $X$, then VI$(X, F)$ admits at least one solution, i.e., $|\Omega| \geq 1$.*

## Proof.

Since $F$ is continuous, $(\mathbf{x}^* - F(\mathbf{x}^*))$ is also continuous. Using the Brouwer's fixed point theorem, the $\mathbf{x}^* = \textbf{proj}_X(\mathbf{x}^* - F(\mathbf{x}^*))$ admits at least one solution. From previous theorem, this solution is also in $\Omega$ □

Remark. The existence of solution can also be established in case of unbounded $X$ and using coercivity of $F$.

# Uniqueness of solution of **VI**$(X, F)$

## Proposition

*If $X$ is a nonempty, closed, and convex set and $F : X \mapsto X$ is strictly monotone function on $X$, then VI$(X, F)$ admits a unique solution, if exists.*

## Proof.

Assume that there exists $\tilde{\mathbf{x}} \neq \mathbf{x}^*$ such that $\tilde{\mathbf{x}}, \mathbf{x}^* \in \Omega$. Since they both are solution to VI$(X, F)$, we have

$$F(\tilde{\mathbf{x}})^T(\mathbf{x} - \tilde{\mathbf{x}}) \geq 0, \forall \mathbf{x} \in X \quad (7)$$

$$F(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X \quad (8)$$

By substituting $\mathbf{x} = \mathbf{x}^*$ in (7) and $\mathbf{x} = \tilde{\mathbf{x}}$ in (8) and adding both yields

$$\left[F(\mathbf{x}^*) - F(\tilde{\mathbf{x}})\right]^T (\mathbf{x}^* - \tilde{\mathbf{x}}) \leq 0 \quad (9)$$

which contradicts the strict monotonicity of $F$. Hence $\tilde{\mathbf{x}} = \mathbf{x}^*$. □

# Nonlinear complementarity problem

Definition (Nonlinear complementarity problem). Let $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continous function. The Non-linear Complementarity Problem (NCP) is to find a vector $\mathbf{x}^* \in \mathbb{R}^n$ such that

$$\begin{aligned} F(\mathbf{x}^*)^T \mathbf{x}^* &= 0 \\ F(\mathbf{x}^*) &\geq 0 \\ \mathbf{x}^* &\geq 0 \end{aligned} \tag{10}$$

# NCP and VI

## Proposition

*NCP and VI$(\mathbb{R}_+^n, F)$ are equivalent, i.e., both problems have precisely the same solution set $\Omega$.*

## Proof.

Let us first assume that $\mathbf{x}^*$ is a solution VI$(\mathbb{R}_+^n, F)$. Then,

- Since $\mathbf{x}^*$ is a solution VI$(\mathbb{R}_+^n, F)$, then $\mathbf{x}^* \in \mathbb{R}_+^n$
- Substituting $\mathbf{x} = \mathbf{x}^* + \mathbf{e}_i$ into (1) yields $(F(\mathbf{x}^*))_i \geq 0$, consequently, $F(\mathbf{x}^*) \geq 0$
- Substitute $\mathbf{x} = 2\mathbf{x}^*$ into (1) yields

$$F(\mathbf{x}^*)^T \mathbf{x}^* \geq 0 \tag{11}$$

  Further, substitute $\mathbf{x} = 0$ into (1), we get

$$F(\mathbf{x}^*)^T \mathbf{x}^* \leq 0 \tag{12}$$

  Together, (11) and (12) implies that $F(\mathbf{x}^*)^T \mathbf{x}^* = 0$.

Conversely, let us assume that $\mathbf{x}^*$ satisfies (10) then $F(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in \mathbb{R}_+^n$ since $F(\mathbf{x}^*)^T \mathbf{x}^* = 0$, $F(\mathbf{x}^*) \geq 0$, and $\mathbf{x} \in \mathbb{R}_+^n$ □

## Suggested reading

- Notes by Prof. Anna Nagurney [Donwload here]
- Patriksson Chapter 3
- BLU book Chapter 3