Introduction to Traffic Assignment

Pramesh Kumar

IIT Delhi

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Transportation Planning

The process of identifying, analyzing, and evaluating transportation infrastructure and systems to improve the built, economic and social environments of communities.[1]

This will help answer questions such as:

- What is the impact on traffic and emissions if
  - a new 6 lane highway corridor is built
  - tolls are introduced on certain links in the network, etc.
- How many highway lanes are required to handle traffic growth after 10 years?
- How many passengers will be riding the new metro line?
- How to compare two or more alternative plans for future transportation network?

We have economic and legal rationale for planning.

[1] Fundamentals of Transportation Wiki

Travel Demand Forecasting

The process used to predict travel behavior and resulting demand for a specific future time frame. We start by dividing the geographical region into Transportation Analysis Zones (TAZs).

1. **Trip Generation** : Whether/when to travel? Estimates the number of trips from/to each zone (also know as trip production and attraction).
   - usually defined as a function of socio-economic, locational, and land-use characteristics of zone
   - divided based on trip purpose (e.g, work versus recreational)
2. **Trip Distribution** : Where to travel (which destination)? Estimates the other end of trips (OD trip matrix).
   - usually defined as a function of trip production and attraction rates and cost of travel between zones.
3. **Mode Choice** : How to travel (which mode)? Estimates the share of each mode from OD trips.
   - proportion of trips by each mode is usually defined as a function of utility of travel by each mode.
4. **Traffic Assignment** : How to travel (which route)?. Estimates traffic flow in transportation network.

Task

Match the following:

1. Traffic assignment	A. Predicts no. of originating from and destined to each zone
2. Trip Distribution	B. Predicts the traffic on each link of the transportation network
3. Mode choice	C. Predicts the number of travelers using each mode
4. Trip generation	D. Estimates the origin-destination flow matrix

Answer

Match the following:

1. Traffic assignment	A. Predicts no. of originating from and destined to each zone
2. Trip Distribution	B. Predicts the traffic on each link of the transportation network
3. Mode choice	C. Predicts the number of travelers using each mode
4. Trip generation	D. Estimates the origin-destination flow matrix

Traffic Assignment

Given the transportation network and the demand, assign the travelers traveling between different zones to the highway network so that you know the number of travelers (flow) and experienced travel time on each link/path in the network?

Circular dependency

The number of travelers taking path π traveling between O-D pair $(r, s) \in Z^2$ depends on the its travel time c^π which further depends on how many travelers are taking that path.

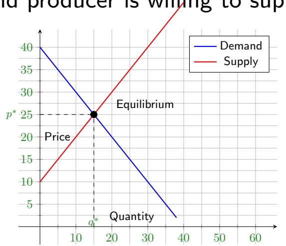
Notion of equilibrium

Classical view of perfectly competitive economic market

Two players

1. Producer
   - behavior characterized by a supply function which expresses the quantity of goods (q) to be produced at a given price (p).
   - as the price increases, it is profitable to produce more quantity.
2. Consumers
   - behavior characterized by a demand function which expresses the quantity of goods consumed (q) at a given price (p).
   - as the price increases, the quantity of consumption decreases.

The point (p^*, q^*) where supply and demand functions meet represents the market clearing or equilibrium point where the consumers are willing to pay p^* to consume quantity q^* and producer is willing to supply quantity q^* at price p^*.



Transportation market

Transportation is a service offered by industry the Government, vehicle manufacturers, transit operators, etc. to travelers.

- Transportation supply is a performance function that describes the level of service (LOS) given the volume (flow) of travelers using it.
  - LOS can be measured in terms of travel time, safety, convenience, comfort, etc.
  - LOS decreases with increase in volume (flow) of travelers.
  - The performance function is usually given as an empirical formula, queuing model, or simulation.
  - The performance function captures the congestion phenonmenon which causes increased delays and costs with increased flow.
- Transportation demand function describes how volume of travelers varies with the level of service offered.
  - Volume increases with increasing LOS.

At equilibrium, there is no incentive for any traveler to behave other than as she actually does. The demand for transportation is equal to its supply at the prevailing cost to the traveler. This course is about computing that equilibrium point.

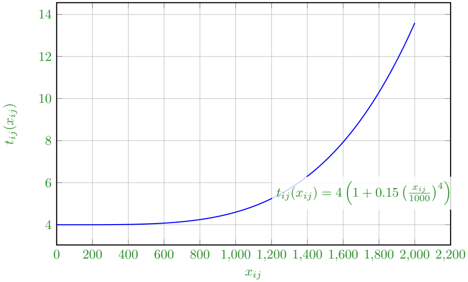
Link performance function

Bureau of Public Roads (BPR) Function

A common link performance (or volume-delay) function

$$t_{ij}(x_{ij}) = t_{ij}^0 \left\{ 1 + \alpha \left(\frac{x_{ij}}{u_{ij}} \right)^{\beta} \right\}$$

where, $t_{ij}^0, u_{ij}, \alpha, \beta$ represent the free flow travel time on the link, capacity of the link and parameters respectively. It is common to use $\alpha = 0.15$ and $\beta = 4$



Game theoretic perspective

The path choice of a traveler depending on the path choices of other travelers in the network can be viewed as a game. Informally, game theory studies interaction among independent and self-interested agents.

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Static games of complete information

Let us define it informally. We need to specify the following:

1. The set of players (agents)
2. For each player, the set of available actions
3. For each player, a payoff (utility) function that depends on the action taken by each player

Gameplay assumptions

1. Timing: Every player chooses an action simultaneously.
2. Rationality: Each player is selfish and trying to maximize their payoff or utility.
3. Common knowledge: (set of players, set of actions, set of payoff functions) is common knowledge.

Task is to find the outcome of the game. With above assumptions, we seek to find the Nash equilibrium of the game in which each player is playing the best action against the other players' actions. In equilibrium, there is no profitable unilateral deviation.

Example: Prisoner's Dilemma

- Two suspects of a crime
- To nail the suspect of the crime, police needs testimony from at least one of the suspects
- Each suspect is interrogated separately
- Each suspect is offered a deal of no sentence that if they confesses on his partner in crime

Game

1. 2 players
2. Actions $= \{\text{confess}, \text{quiet}\}$
3. Payoff matrix

		P2 C	P2 Q
P1	C	-5, -5	0, -10
P1	Q	-10, 0	-1, -1

In equilibrium, both of them confess even though they are better off being quiet.

Transportation routing game

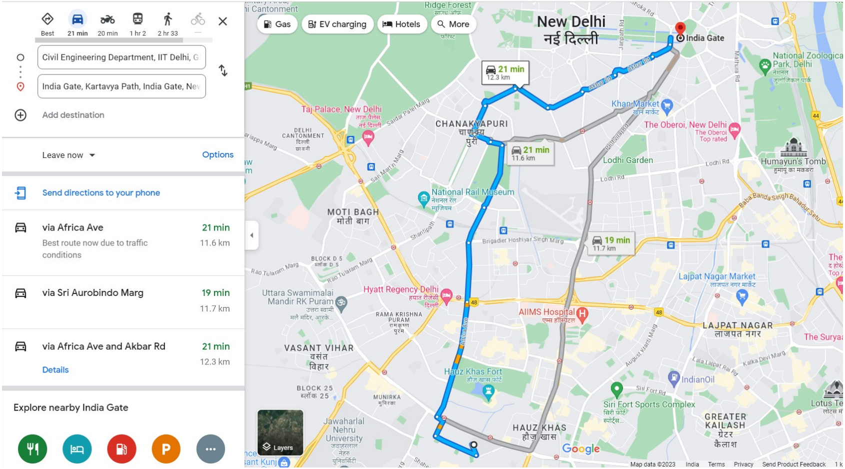


Figure: Which route would you take from Civil & Environmental Engineering to India Gate?

Transportation routing game

1. Players are the travelers
2. For each player, the actions are available paths between each O-D pair
3. For each player, a payoff (utility) function is the path travel time.

As rational players, each traveler seek to minimize their travel time.

Wardrop equilibirum

Assumptions:

- Travelers choose the path with minimum travel time
- Travelers have complete and accurate information about transportation network

Outcome:

User Equilibrium (Wardrop equilibrium)

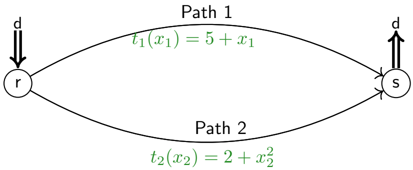
No user can decrease his/her own travel time by unilaterally changing the path. In traffic networks, this leads to every used path connecting an origin and destination having equal and minimal travel time.

Remark. UE flows should be thought of as the steady state evolving after number of periods have passed and travelers have adjusted to paths based on prevailing conditions.

Example

Three possible cases:

1. All users take path 1
   $t_1(d) < t_2(d)$
2. All users take path 2
   $t_1(d) > t_2(d)$
3. Some users take path 1 and some take path 2
   $t_1(x_1) = t_2(x_2)$



- Let $d = 1 \implies t_1(1) = 6, t_2(1) = 3 \implies x_1^* = 0, x_2^* = 1$
  All users take path 2 ($t_1^* = 5, t_2^*(1) = 3$)
- Let $d = 2 \implies t_1(2) = 7, t_2(2) = 6$, but then $t_1 = 5$.
  Some users take path 1 and some take path 2
  $t_1(x_1^*) = t_2(x_2^*); x_1^* + x_2^* = 2$
  $\implies x_2^* = 1.791, x_1^* = 0.209$ and $t_1^* = t_2^* = 5.21$

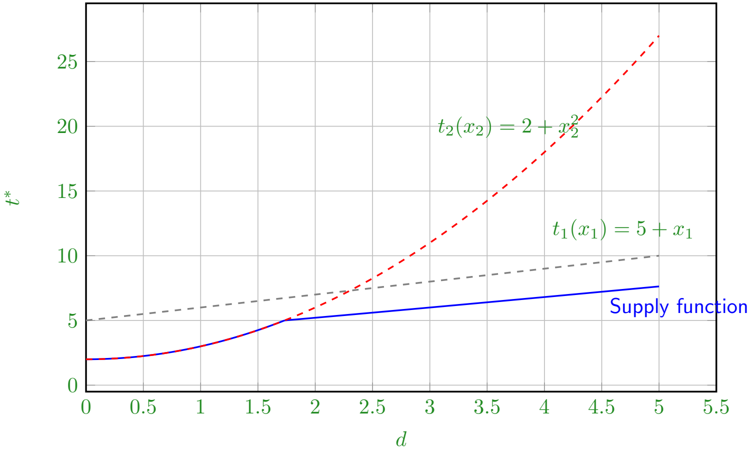


Figure: O-D performance function

Steps for finding UE solution using its definition[2]:

1. Select a set of paths that are likely to be used according to their free flow travel time. A safe assumption is to select one path for each OD pair.
2. Write the equations for UE conditions, i.e. equal path travel times for the selected paths in each OD pair.
3. Write the equations for conservation of demand for each OD pair.
4. Solve the system of equations to calculate link flows $\mathbf{x}$ and then calculate link travel times $\mathbf{t}$.
5. If the solution satisfies the UE conditions, stop.
6. Otherwise, modify the selected set of paths by adding other likely used paths or removing unused paths and go to step 2.

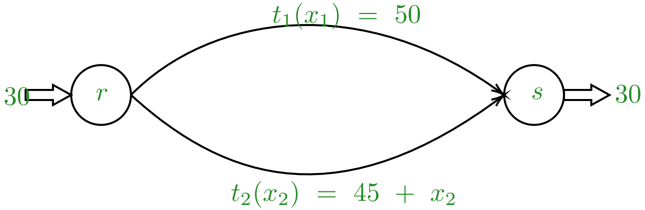
Definition (Total system travel time (TSTT)). A useful metric that measures how many veh-hours travelers are spending in the system. It is defined as:

$$\boxed{TSTT(\mathbf{x}) = \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij})} \tag{1}$$

[2] Taken from CEGE 8217 taught by Prof. Alireza Khani

Example 1[3]

Evaluate UE in the following network. Upper link may represent dedicated transit line and the lower link may represent a highway.



$x_1^* = 25, x_2^* = 5$
$t_1^*(x_1^*) = t_2^*(x_2^*) = 50$

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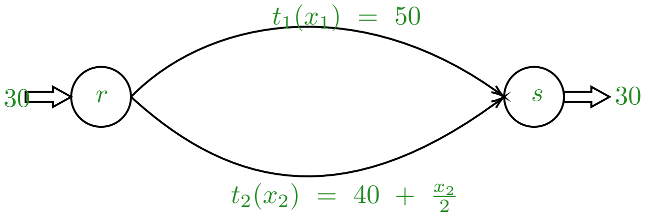
[3] Taken from BLU book Chapter 5

Example 1

Let's try to improve the lower link by changing its travel time function to $t_2(x_2) = 40 + \frac{x_2}{2}$. Evaluate the travel times in UE solution now.

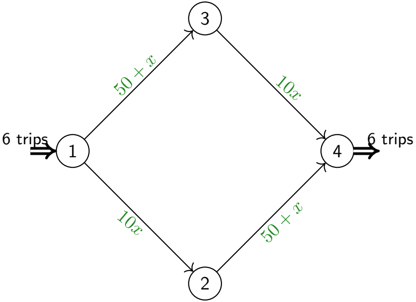


$x_1^* = 10, x_2^* = 20$
$t_1^*(x_1^*) = t_2^*(x_2^*) = 50$ But the travel times remain 50 even though bottom link was improved. The effect of improvement was taken away by some travelers shifting from the upper link to the lower link. This means that improving a link may not lead to reduced travel time. This paradox is popularly known as Pigou-Knight-Downs paradox .

Example 2

Task

Find UE. What is the total system travel time (veh-hrs) spent?



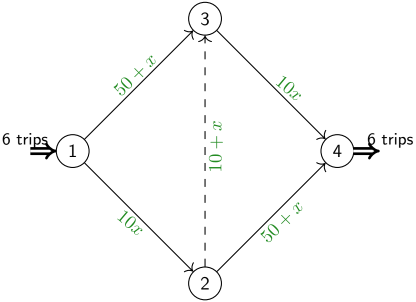
$h^{\pi^1} = h^{\pi^2} = 3; c^{\pi^1} = c^{\pi^2} = 83$

$TSTT = 83.3 + 83.3 = 496$ veh-min.

Example 2

Task

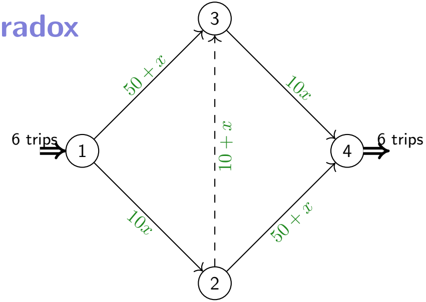
We try to improve congestion by building a bridge. Compute the total system travel time now.



Braess' paradox

Task

We try to improve congestion by building a bridge. Compute the total system travel time now.



$h^{\pi^1} = h^{\pi^2} = h^{\pi^3} = 2; c^{\pi^1} = c^{\pi^2} = c^{\pi^3} = 92$

$TSTT = 92.2 + 92.2 + 92.2 = 552$ veh-min.

The situation actually got worse.

This is known as Braess' paradox! Adding a new link may increase the travel time of commuters.

This happens because of the selfish behavior of commuters.

Braess' paradox can be observed in springs!

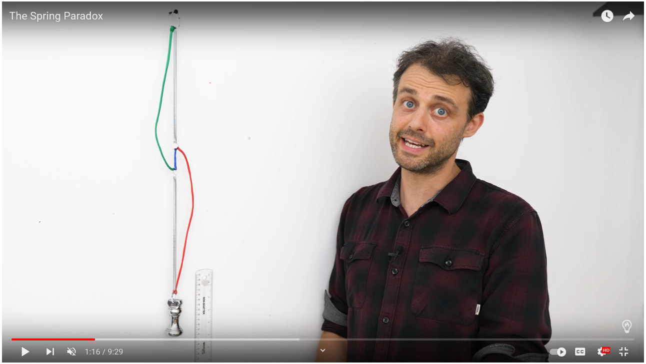


Figure: https://www.youtube.com/watch?v=Cg73j3QYRJc

Questions: In UE, every traveler doing what is best for themselves i.e., minimizing their individual travel time. Does it mean it will result in the best outcome for the system? Does it minimize the overall veh-hrs spent in the system? Is there any other way of routing travelers in the network?

System optimal traffic assignment and Price of Anarchy

Definition (System optimal (SO) traffic assignment).: Given the transportation network and the demand, assign the travelers traveling between different zones to the highway network so as to minimize the total system travel time in the network.

Remark. In SO assignment, some travelers belonging to an O-D pair may face higher travel time than others.

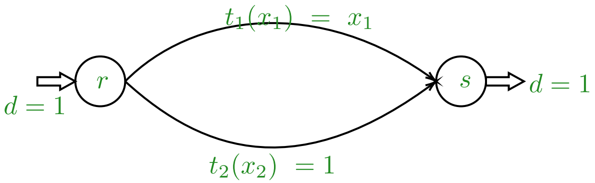
Remark. The UE assignment is sometimes referred to as descriptive assignment and the SO assignment is sometimes referred to as prescriptive assignment.

Definition (Price of Anarchy (ρ):).
The ratio of TSTT obtained from the UE assignment to the SO assignment is called the *Price of Anarchy*, which is always greater than or equal to 1.

$$\boxed{\rho = \frac{TSTT(\mathbf{x}_{UE})}{TSTT(\mathbf{x}_{SO})}}$$

This measure tells us how bad is selfish routing for the network.

Example 3: Pigou (1920)



- In UE, $x_1^* = 1, x_2^* = 0$, $TSTT(\mathbf{x}_{UE}) = 1$
- In SO, we minimize $x_1 \times x_1 + (1 - x_1) \times 1$. $x_1 = x_2 = \frac{1}{2}$. $TSTT(\mathbf{x}_{SO}) = \frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{3}{4}$
- $\rho = \frac{TSTT(\mathbf{x}_{UE})}{TSTT(\mathbf{x}_{SO})} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

Theorem (Roughgarden (2002))

Let $\mathcal{T}$ be an arbitrary set of nonnegative, continuous, and nondecreasing cost functions in a Pigou-like network. Define $\rho(\mathcal{T})$ as the worst-case Price of Anarchy with cost functions in $\mathcal{T}$. Then,

$$\rho(\mathcal{T}) = \sup_{t \in \mathcal{T}} \sup_{d \geq 0} \sup_{x \geq 0} \left\{ \frac{d \times t(d)}{x \times t(x) + (d - x) \times t(d)} \right\} \tag{2}$$

Description	Typical Representative	Price of Anarchy
Linear	$ax + b$	$4/3$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.6$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.9$
Quartic	$ax^4 + \cdots$	$\frac{5\sqrt[4]{5}}{5\sqrt[4]{5}-4} \approx 2.2$
Polynomials of degree $\leq d$	$\sum_{i=0}^{d} a_i x^i$	$\frac{(d+1)\sqrt[d]{d+1}}{(d+1)\sqrt[d]{d+1}-d} \approx \frac{d}{\ln d}$

Figure: Worst-case ρ in networks with different cost functions [4]

[4] Taken from CS364A Lecture 11 taught by Prof. Tim Roughgarden

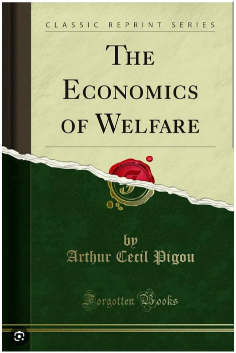
Critique of static traffic assignment

- Static assignment does not capture how congestion evolves over time.
  - Travelers departing during peak hours may experience higher travel time than travelers who are departing during non-peak hours.
  - Travelers may take different paths during different times of the day.
- All vehicles belonging to a link are assumed to experience equal travel on that link even though some travelers may arrive late.
- Link performance functions such as BPR function can take flow greater than the capacity which is not realistic.
- There may be queue spillback.

Suggested reading

1. BLU Chapter 1 and 5
2. Sheffi Chapter 1
3. Patriksson Chapter 1 and 2

Pigou (1920)



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the other sources.[1] This result is of considerable theoretical importance, because it is in direct conflict with the widespread opinion that, apart from certain possible indirect effects, differential taxes are necessarily wasteful and necessarily cause people to obtain what they want by a more costly, instead of by a less costly, route. This opinion is incorrect, and the nature of the error can be easily illustrated. Suppose there are two roads ABD and ACD both leading from A to D. If left to itself, traffic would be so distributed that the trouble involved in driving a "representative" cart along each of the two roads would be equal. But, in some circumstances, it would be possible, by shifting a few carts from route B to route C, greatly to lessen the trouble of driving those still left on B, while only slightly increasing the trouble of driving along C. In these circumstances a rightly chosen measure of differential taxation against road B would create an "artificial" situation superior to the "natural" one. But the measure of differentiation must be rightly chosen.

Wardrop (1952)

WARDROP ON SOME THEORETICAL ASPECTS OF ROAD TRAFFIC RESEARCH 325

ROAD ENGINEERING DIVISION MEETING

24 January, 1952

Brigadier A. C. HUGHES, C.B.E., T.D., B.Sc., M.I.C.E.. Chairman of the Division, in the Chair

The following Paper was presented for discussion and, on the motion of the Chairman, the thanks of the Division were accorded to the Author.

Road Paper No. 36

"Some Theoretical Aspects of Road Traffic Research" *

by

John Glen Wardrop, B.A.

OF ROAD TRAFFIC RESEARCH 345

(1) The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

(2) The average journey time is a minimum.

The first criterion is quite a likely one in practice, since it might be assumed that traffic will tend to settle down into an equilibrium situation in which no driver can reduce his journey time by choosing a new route. On the other hand, the second criterion is the most efficient in the sense that it minimizes the vehicle-hours spent on the journey. In practice, of course, drivers will be influenced by other factors, such as the state of the roads, and the comfort or discomfort of driving in general. However, it is clearly difficult to allow for these psychological factors.

Consider each criterion separately.

Braess (1968)

In German

Über ein Paradoxon aus der Verkehrsplanung

Von D. BRAESS, Münster [1])

Eingegangen am 28. März 1968

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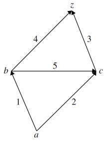
English Translation

On a Paradox of Traffic Planning

Dietrich Braess
Faculty of Mathematics, Ruhr-University Bochum, 44780 Bochum, Germany, dietrich.braess@rub.de

Anna Nagurney, Tina Wakolbinger
Department of Finance and Operations Management, Isenberg School of Management, University of Massachusetts, Amherst, Massachusetts 01003 {nagurney@gbfin.umass.edu, wakolbinger@som.umass.edu}



$$t_1(\varphi) = t_3(\varphi) = 10\varphi,$$
$$t_2(\varphi) = t_4(\varphi) = 50 + \varphi,$$
$$t_5(\varphi) = 10 + \varphi.$$

This means that for real-life traffic practice: *In unfavorable situations an extension of the road network may lead to increased travel times.*

Thank You![5]

[5] https://www.thenewsminute.com/