Elementary definitions in Graph Theory

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Outline

Introduction

Definition (Network). A network is interconnection among set of items. Examples include internet network, social network, airline networks, highway networks, etc.

Transportation networks

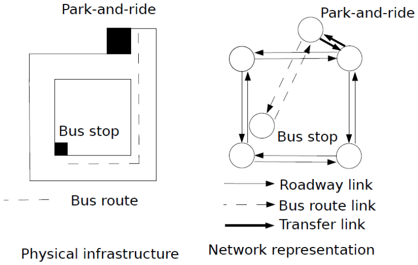


Figure 1.1: Nodes and links in transportation networks.

Table 1.1: Nodes and links in different kinds of transportation networks.

Network type	Nodes	Links
Roadway	Intersections	Street segments
Public transit	Bus or train stops	Route segments
Freight	Factories, warehouses, retailers	Shipping options
Air	Airports	Flights
Maritime	Ports	Shipping channels

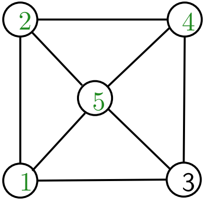
Figure: Credits: BLU Book

Outline

Undirected graph

Definition (Undirected graph/network). An undirected graph G is a pair (N, A), where N is the set of nodes and A is the set of links whose elements are unordered pair of distinct nodes.

Example(s). $N = \{1, 2, 3, 4, 5\}$,
$A = \{(1, 2), (1, 3), (1, 5), (5, 4), (5, 3), (5, 2), (2, 4), (3, 4)\}$

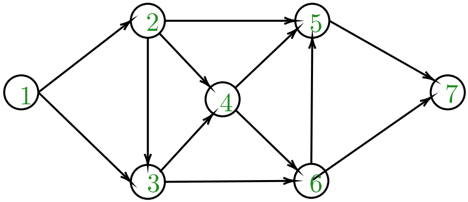


Remark. Let $|N| = n$. Then, $|E| = m \leq \frac{n(n-1)}{2}$.

Directed graph

Definition (Directed network/graph). A directed graph is pair (N, A), where N denotes the set of nodes/vertices and $A \subseteq N \times N$ denotes the set of links/edges/arcs whose elements are ordered pair of distinct nodes.

Example(s). $N = \{1, 2, 3, 4, 5, 6, 7\}$
$A = \{(1,2), (1,3), (2,3), (2,4), (2,5), (3,4), (3,6), (4,5), (4,6), (5,7), (6,5), (6,7)\}$



Definition (). If $e = (i, j) \in A$, then

1. i and j are endpoints of e.
2. i is the tail node and j is the head node of e.
3. (i, j) emanates from i and terminates at node j.
4. (i, j) is incident to nodes i and j.
5. (i, j) is outgoing link of node i and incoming link of node j.

Definition (Degree). The number of incoming and outgoing links of a node $i \in N$ are called indegree and outdegree respectively. The sum of indegree and outegree is called degree.

Definition (Multilinks). Two or more links with same head and tail nodes.

Definition (Loop). A link whose tail and head nodes are the same.

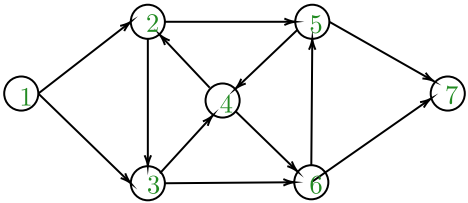
Note: In this course, we assume that graphs contain no loops or multiarcs.

Definition (Subgraph). A graph $G^{'}(N^{'}, A^{'})$ is a subgraph of $G(N, A)$ if $N^{'} \subseteq N$ and $A^{'} \subseteq A$. A subgraph $G^{'}(N^{'}, A^{'})$ of $G(N, A)$ is said to be induced by $N^{'}$ if $A^{'}$ contains links with their end points in $N^{'}$.

Definition (Walk). A collection of links $W = \{(u_1, v_1), \cdots, (u_q, v_q)\}$ is an $s-t$ walk if

1. $u_1 = s$
2. $v_i = u_{i+1}, \forall i = 1, ..., q-1$
3. $v_q = t$

Example(s).



$W_1 = \{(1,2),(2,5),(5,7)\}$,
$W_2 = \{(1,2),(2,3),(3,4),(4,2),(2,5),(5,7)\}$,
$W_3 = \{(1,3),(3,6),(6,5),(5,4),(4,6),(6,7)\}$
are all exmples of $1-7$ walks.

Definition (Path). An $s-t$ path is an $s-t$ walk without any repeated nodes.
In above example, W_1 is a $1-7$ path while W_2 and W_3 are not.

Definition (Cycle). A cycle is a path with same first and last nodes.

Definition (Tour). A tour is a cycle including all nodes of the graph.

Definition (Acyclic graph). A graph without any cycles is acyclic.

Definition ().

1. Nodes $i \in N$ and $j \in N$ are said to be connected if there exists at least one path between i and j.
2. A graph is said to be connected graph if every pair of its nodes are connected. Otherwise, the graph is called disconnected.

Definition (Tree). A tree is a connected graph that contains no cycles.

Proposition

1. *A tree on n nodes contains exactly $n-1$ links.*
2. *A tree has at least 2 leaf nodes (i.e., nodes with degree 1).*
3. *Every pair of nodes are connected by a unique path.*

Proof.

1. (Proof by induction) Let $P(n)$ be the statement that a tree on n nodes contains exactly $n-1$ links. $P(1) = 0$ since there is only one node and a link requires at least two nodes. Let us assume that $P(k)$ is true, i.e., a tree on k nodes contains exactly $k-1$ links. Then, we can add another node to this graph with one link and that would still be a tree with k links, which means that $P(k+1)$ is true.
2. Assuming $n < \infty$, we prove this by contradiction. Assume that a tree on n nodes has only one leaf u. Then, find the longest path from u in the tree. The longest path cannot end at u because that is not a path but cycle. Let us assume that it ends at v. If v has degree 1 then we are done. If it has degree 2, then it is not a longest path.
3. Proof by induction. Not possible to add another node without creating a cycle.

□

Outline

Network representation

- The performance of a network algorithm depends not only on the algorithm but also on which data structure we use to store the network.
- We need to store how nodes are connected as well as capacities or costs associated to links.

Data structures

1. Node-link incidence matrix
2. Node-node adjacency matrix
3. Adjacency list
4. Forward (Backward) Star

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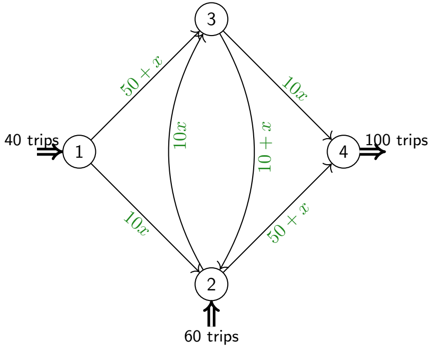
Highway networks

A few notations:

- A highway network is represented by a directed graph $G(N, A)$
  - *Nodes*: intersections or locations where road characteristics change (intersections, freeway merges/diverges)
  - *Links*: roadways connecting nodes
  - *Commodities*: travelers (cars, motorcycles, etc.)
- $Z \subseteq N$ is the set of zones (this actually represents the centroid of traffic analysis zone where we'll assume any trip starts or ends)
- $D = \{d^{rs}\}_{(r,s)\in Z^2}$: A matrix having between different origin-destination pairs
- x_{ij} : flow or volume on link $(i, j) \in A$ during the analysis time period
- t_{ij}: travel time on link $(i, j) \in A$, usually function of x_{ij}
- Π^{rs}: set of paths between origin-destination pair $(r, s) \in Z^2$
- h^{π}: flow or volume on path $\pi \in \Pi$
- c^{π}: travel time on path $\pi \in \Pi$
- $\delta^{\pi}_{ij} = 1, \text{ if } (i, j) \in \pi, 0, \text{ otherwise}$ (Link-path incidence matrix)

Example

Write down N, A, d, Π. Assume that demand is equally distributed among paths, compute the value of $\mathbf{x}, \mathbf{c}$.

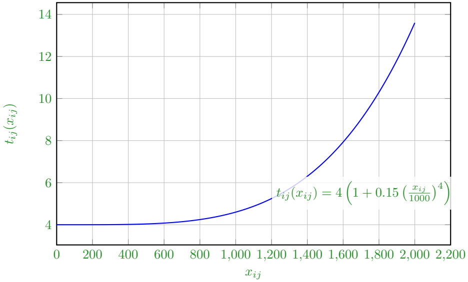


Bureau of Public Roads (BPR) Function

A common link performance (or volume-delay) function

$$t_{ij}(x_{ij}) = t_{ij}^0 \left\{ 1 + \alpha \left(\frac{x_{ij}}{u_{ij}} \right)^\beta \right\}$$

where, $t_{ij}^0, u_{ij}, \alpha, \beta$ represent the free flow travel time on the link, capacity of the link and parameters respectively.



Thank you!