Traffic assignment with elastic demand

Pramesh Kumar

IIT Delhi

October 15, 2025

Motivation

We assumed that demand between any O-D pair is fixed. However,

- It may be influenced by the level of service on the network.
- For example, as the congestion worsens, travelers may shift to different mode, change the departure time of their travel, or even decide not to travel (work from home).
- They may be induced demand phenomenon where major expansion of highway may result in increase in the travel demand.

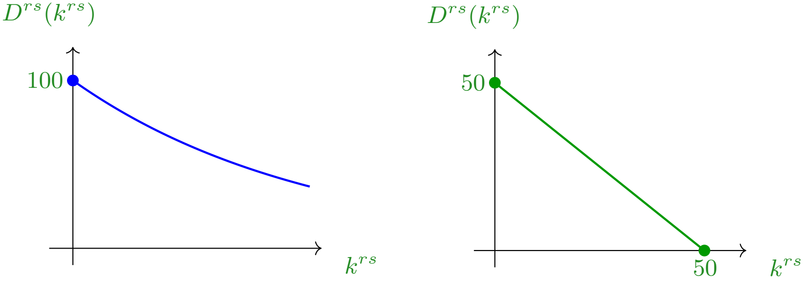
In order to model these phenomenon, we need to relax the assumption that OD matrix is fixed and known. For this purpose, we can assume demand between any O-D pair $(r, s) \in Z^2$ to be a function of minimum travel time between r and s.

Demand function

We assume demand between any O-D pair $(r, s) \in Z^2$ to be a function of minimum travel time between r and s.

$$\boxed{d^{rs} = D_{rs}(k^{rs})}$$

where, k^{rs} is the minimum travel time between r and s.



(a) $D^{rs}(k^{rs}) = 100e^{-0.01k^{rs}}$

(b) $D^{rs}(k^{rs}) = \max\{50 - k^{rs}, 0\}$

Figure: Examples of elastic demand function

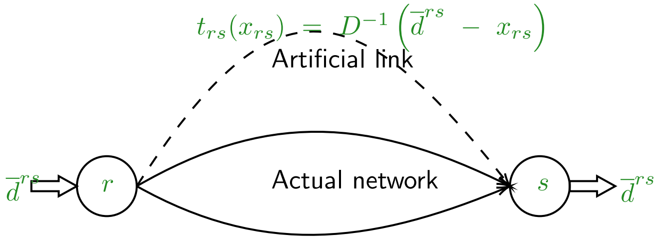
Demand function

- The demand function can include O-D specific parameters reflecting population size, employment opportunities, income distributions, etc. For example, $d^{rs} = A_r B_s f(k^{rs})$, where A_r and B_s are the parameters specific to origin r and destination s.
- The demand function is expected to be non-increasing function of k^{rs}, i.e., when k^{rs} reduces, d^{rs} increases and vice-versa. We assume that it is strictly decreasing which will help us take its inverse.
- The demand function is also expected to be bounded from above, i.e., $d^{rs} \leq \bar{d}^{rs}, \forall (r, s) \in Z^2$. Due to limited population and limited duration of analysis period, the number of trips should be bounded from above.

Gartner's transformation

With a small trick, we can convert the traffic assignment problem with elastic demand (TAP-E) into a traffic assignment problem with fixed demand (TAP).

- Create an artificial link between each origin-destination pair with travel time function $t_{rs}(x_{rs}) = D_{rs}^{-1}(\bar{d}^{rs} - x_{rs})$, where x_{rs} is the flow on the artificial link (r, s).
- As we assumed that demand function is strictly decreasing, the inverse of the demand function will be strictly increasing function of its flow x_{rs} making it the legitimate link performance function.



Gartner's transformation

- In the transformed network, the flow on the artificial link represents the number of travelers who chose not to travel due to excess congestion.
- In equilibrium, all used path will have equal and minimal travel time k^{rs}. If the artificial link is used, then $k^{rs} = D_{rs}^{-1}(\bar{d}^{rs} - x_{rs}) \implies x_{rs} = \bar{d}^{rs} - D_{rs}(k^{rs})$,which is exactly the number of travelers who chose not to travel when equilibrium travel times are k^{rs}.
- Although Gartner's transformation makes the TAP-E easy to solve using existing solution algorithms for TAP but it results in excessive number of links in the network as an artificial link needs to be created for each O-D pair.

Variational inequality formulation of TAP-E

The equilibrium solution $(\mathbf{x}^*, \mathbf{d}^*)$ will satisfy the following variational inequality.

$$\boxed{\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) - \mathbf{D}^{-1}(\mathbf{d}^*)^T(\mathbf{d} - \mathbf{d}^*) \geq 0, \forall(\mathbf{x}, \mathbf{d})} \tag{1}$$

Proposition

Under the assumptions, $\mathbf{D}$ *are strictly decreasing,* $\mathbf{t}$ *are strictly increasing, and feasible space is non-empty, closed and convex set, then the variational inequality problem* (1) *admits unique solution of link flows and demand values.*

Optimization formulation of TAP-E

Due to monotonicity assumptions, the above VI can be formulated as the following convex optimization problem.

$$\underset{\mathbf{x},\mathbf{h},\mathbf{d}}{\text{minimize}} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx - \sum_{(r,s)\in Z^2} \int_0^{d^{rs}} D_{rs}^{-1}(x)dx \tag{2a}$$

$$\text{subject to} \quad \sum_{\pi\in\Pi^{rs}} h^\pi = d^{rs}, \forall (r,s) \in Z^2 \tag{2b}$$

$$h^\pi \geq 0, \forall \pi \in \Pi \tag{2c}$$

$$d^{rs} \geq 0, \forall (r,s) \in Z^2 \tag{2d}$$

$$x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^\pi h^\pi, \forall (i,j) \in A \tag{2e}$$

KKT conditions of optimization formulation of TAP-E

Theorem

The KKT conditions of the optimization formulation (2) *corresponds to the following UE conditions, i.e.,*

1. *Any used path for a given O-D pair must have minimal travel time. If a path is not used by travelers, then the travel time on this path must be greater or equal to the minimal path travel time.*
2. *If the demand between any O-D pair is positive, then it must be given by the demand function of minimum path travel time. Further, if the demand is zero then the minimum travel time path is greater than threshold travel time at which any travel happens.*

Proof.

Let's do it together. □

Solution algorithm for solving TAP-E

- Frank-Wolfe algorithm can be used to solve (2).
- In this formulation, d^{rs} is also a decision variable.
- We need to maintain both auxiliary link flows $\{y_{ij}\}_{(i,j)\in A}$ and auxiliary demand values $\{d^{rs}\}_{(r,s)\in Z^2}$
- For evaluating the value of step size, we can solve the following equation.

$$\boxed{\sum_{(i,j)\in A} t_{ij}(\lambda y_{ij} + (1-\lambda)x_{ij})(y_{ij} - x_{ij}) - \sum_{(r,s)\in Z^2} D_{rs}^{-1}(\lambda \hat{d}^{rs} + (1-\lambda)d^{rs})(\hat{d}^{rs} - d^{rs}) = 0} \tag{3}$$

- We need to two gap values for checking the convergence. $\text{gap}_1 = \frac{\text{TSTT}-\text{SPTT}}{\text{SPTT}}$ and $\text{gap}_2 = \frac{\sum_{(r,s)\in Z^2} |d^{rs} - D_{rs}(k^{rs})|}{\sum_{(r,s)\in Z^2} D_{rs}(k^{rs})}$

Frank-Wolfe algorithm for solving TAP-E

procedure $\mathrm{FW}(G, \mathbf{t}, \mathbf{D}, \mathsf{tol}_1, \mathsf{tol}_2)$
    $n = 1$, Choose some initial value of demand $(d^{rs})^1$ and link flows x_{ij}^1
    $\mathsf{gap}_1 = \mathsf{gap}_2 = \infty$ ▷ Initialization
    while $\mathsf{gap}_1 > \mathsf{tol}_1$ **and** $\mathsf{gap}_2 > \mathsf{tol}_2$ **do**
        $y_{ij}^k = 0, \forall (i,j) \in A$ ▷ Auxiliary flows
        Find the shortest path travel time k^{rs} each O-D pair $(r,s) \in Z^2$
        Calculate the auxiliary demand $(\hat{d}^{rs})^n = D_{rs}(k^{rs})$ for each O-D pair $(r,s) \in Z^2$
        Calculate the auxiliary $\{y_{ij}^n\}_{(i,j)\in A}$ link flows by loading the auxiliary demand $\hat{d}^{rs}$ onto shortest paths.
        Evaluate λ using (3)
        for $(i,j) \in A$ **do**
            $x_{ij}^{n+1} \leftarrow (1-\lambda)x_{ij}^n + \lambda y_{ij}^n$ ▷ Update link flows
            $(d^{rs})^{n+1} \leftarrow (1-\lambda)(d^{rs})^n + \lambda(\hat{d}^{rs})^n$ ▷ Update demand
            $t_{ij}^{k+1} \leftarrow t_{ij}(x_{ij}^{k+1})$ ▷ Update link travel times
        end for
        Evaluate gap_1 and gap_2
        $k \leftarrow k+1$
    end while
end procedure

System Optimal assignment with elastic demand

$$
\begin{aligned}
Z^{SO} = \underset{\mathbf{h},\mathbf{x}}{\text{minimize}} \quad & \sum_{(i,j)\in A} x_{ij}t_{ij}(x_{ij}) - \sum_{(r,s)\in Z^2} \int_0^{d^{rs}} D_{rs}^{-1}(x)dx && \text{(4a)}\\
\text{subject to} \quad & \sum_{\pi\in\Pi^{rs}} h^{\pi} = d^{rs}, \forall (r,s) \in Z^2 && \text{(4b)}\\
& h^{\pi} \geq 0, \forall \pi \in \Pi && \text{(4c)}\\
& x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^{\pi} h^{\pi}, \forall (i,j) \in A && \text{(4d)}
\end{aligned}
$$

In this formulation, we can equivalently write the objective function (4a) as

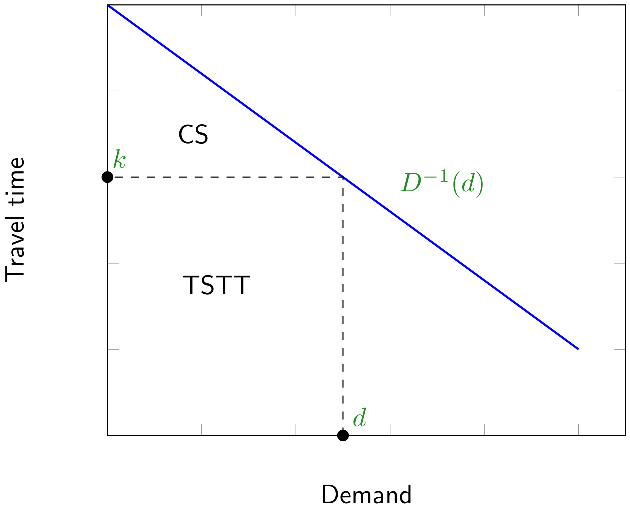
$$
\begin{aligned}
& \sum_{(r,s)\in Z^2} d^{rs}k^{rs} - \sum_{(r,s)\in Z^2} \int_0^{d^{rs}} D_{rs}^{-1}(x)dx && \text{(5)}\\
= & \sum_{(r,s)\in Z^2} \left(d^{rs}k^{rs} - \int_0^{d^{rs}} D_{rs}^{-1}(x)dx \right) && \text{(6)}
\end{aligned}
$$

(6) represents the negative of consumer surplus (CS) which is a fundamental concept in microeconomics. SO with elastic demand can be solved by solving the UE formulation with link travel time functions being replaced by $\tilde{t}_{ij}(x) = t_{ij}(x) + x_{ij}\frac{dt_{ij}(x)}{dx}$.

Consumer surplus

Each driver has a certain threshold travel time for making a trip. If the travel time is less than this threshold, the driver will make the trip. The demand function captures the thresholds of all the drivers, i.e., it represents how many travelers will travel if the shortest path travel time is k^{rs}. Consumer surplus represents the extra "utility" (or "benefit") every driver earns, when they make a trip. For example, for a driver if the threshold was 20 minutes but the shortest path travel time is 15 minutes, then the driver earned a benefit of 5 minutes by making this trip. The objective function (6) aggregates the consumer surplus of all the drivers.

Consumer surplus



Final remarks

- Including elastic demand in the traffic assignment problem is definitely a step towards more realism
- However, the calibration of the demand functions for each O-D pair can be a tedious task.
- The idea is strong and can help understand traffic assignment with mode choice (see homework problem).

Thank you!