Mathematical Modeling of User Equilibrium

Pramesh Kumar

IIT Delhi

September 3, 2025

Notations

- A highway network is represented by a directed graph $G(N, A)$, where N is the set of nodes and A is the set of links.
- $Z \subseteq N$ is the set of zones (this actually represents the centroid of traffic analysis zone where we'll assume any trip starts or ends)
- $D = \{d^{rs}\}_{(r,s) \in Z^2}$: A matrix having between different origin-destination pairs
- x_{ij} : flow or volume on link $(i, j) \in A$ during the analysis time period
- x_{ij}^{rs} : flow or volume on link $(i, j) \in A$ traveling between $(r, s) \in Z^2$
- t_{ij}: travel time on link $(i, j) \in A$, usually function of x_{ij}
- Π^{rs}: set of paths between origin-destination pair $(r, s) \in Z^2$
- $\Pi = \cup_{(r,s) \in Z^2} \Pi^{rs}$: set of all paths in the network
- h^{π}: flow or volume on path $\pi \in \Pi$
- c^{π}: travel time on path $\pi \in \Pi$
- $\delta_{ij}^{\pi} = 1, \text{ if } (i, j) \in \pi, 0, \text{ otherwise}$
- $\Delta = \{\delta_{ij}^{\pi}\}_{(i,j) \in A, \pi \in \Pi}$: link-path incidence matrix

Feasible assignment

In terms of paths flows, an assignment $\{h^\pi\}_{\pi \in \Pi}$ is defined as the feasible assignment if it satisfies the following constraints:

$$h^\pi \geq 0, \forall \pi \in \Pi \qquad \text{(non-negative path flows)} \tag{1}$$

$$\sum_{\pi \in \Pi^{rs}} h^\pi = d^{rs}, \forall rs \in Z^2 \qquad \text{(conservation of flow)} \tag{2}$$

Let's denote $H = \left\{ \mathbf{h} \in \mathbb{R}^{|\Pi|} \middle| \ (1) - (2) \right\}$ as the set of feasible path flows.

In terms of link flows, an assignment $\{x_{ij}\}_{(i,j) \in A}$ is defined as the feasible assignment if it satisfies the following constraints:

$$x_{ij}^{rs} \geq 0, \forall (i,j) \in A \qquad \text{(non-negative link flows)} \tag{3}$$

$$\sum_{j \in FS(i)} x_{ij}^{rs} - \sum_{j \in BS(i)} x_{ji}^{rs} = \begin{cases} d^{rs}, & \text{if } i = r \\ -d^{rs}, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases} , \forall i \in N, \forall (r,s) \in Z^2 \tag{4}$$

$$x_{ij} = \sum_{(r,s) \in Z^2} x_{ij}^{rs}, \forall (i,j) \in A \qquad \text{(conservation of flow)} \tag{5}$$

Feasible assignment

We can also aggregate these with respect to destinations/origins only.
Let $\{x_{ij}^s\}$ be the flow on link $(i,j) \in A$ going to destination $s \in Z$.

$$x_{ij}^s \geq 0, \forall (i,j) \in A, \forall s \in Z \qquad \text{(non-negative link flows)} \quad (6)$$

$$\sum_{j \in FS(i)} x_{ij}^s - \sum_{j \in BS(i)} x_{ji}^s = \begin{cases} d^{rs}, & \text{if } i = r \\ -\sum_{r \in Z} d^{rs}, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases}, \forall i \in N, \forall (r,s) \in Z^2 \quad (7)$$

$$x_{ij} = \sum_{s \in Z} x_{ij}^s, \forall (i,j) \in A \qquad \text{(conservation of flow)} \quad (8)$$

Let's denote $X = \left\{ \mathbf{x} \in \mathbb{R}^{|A|} \middle| \text{ (3)} - \text{(5)} \right\}$ as the set of feasible link flows.

Link-path relations

The link flows can be obtained from path flows using the following relation:

$$x_{ij} = \sum_{(r,s)\in Z^2} \sum_{\pi\in\Pi^{rs}} \delta_{ij}^{\pi} h^{\pi} \tag{9}$$

In vector-matrix notations,

$$\mathbf{x} = \Delta\mathbf{h} \tag{10}$$

Similarly, the path travel times can be obtained from link travel times using the following relation

$$c^{\pi} = \sum_{(i,j)\in A} \delta_{ij}^{\pi} t_{ij} \tag{11}$$

In vector-matrix notations,

$$\mathbf{c} = \Delta^{T}\mathbf{t} \tag{12}$$

Flow Decomposition Theorem

Theorem (AMO Chapter 3 , Th. 3.5)

Every path and cycle flow has a unique representation as non-negative link flows. Conversely, every non-negative link flows $\mathbf{x}$ *can be represented as a path and cycle flow (though not necessarily uniquely).*

Proposition

H is compact (closed & bounded) and convex.

Proof.

- *Closed*: Consider a sequence $\{\mathbf{h}^1, \mathbf{h}^2, \cdots\}$ such that $\mathbf{h}^k \in H, \forall k$. As $k \mapsto \infty$, $\mathbf{h}_k$ converges to $\mathbf{h} \in H$.
- *Bounded*: Since we have $h^\pi \leq \max_{(r,s)\in Z^2} d^{rs}, \forall \pi \in \Pi$, we have $\|\mathbf{h}\| \leq \sqrt{|\Pi|}(\max_{(r,s)\in Z^2} d^{rs})$.
- *Convex*: Consider $\mathbf{h}_1, \mathbf{h}_2 \in H$. For any $\lambda \in [0,1]$, $\lambda\mathbf{h}_1 + (1-\lambda)\mathbf{h}_2 \geq 0$ since $\mathbf{h}_1, \mathbf{h}_2 \geq 0$. Also, $\sum_{\pi\in\Pi^{rs}}(\lambda\mathbf{h}_1^\pi + (1-\lambda)\mathbf{h}_2^\pi) = \lambda d^{rs} + (1-\lambda)d^{rs} = d^{rs}, \forall rs \in Z^2$. Therefore, $\lambda\mathbf{h}_1 + (1-\lambda)\mathbf{h}_2 \in H$.

□

Proposition

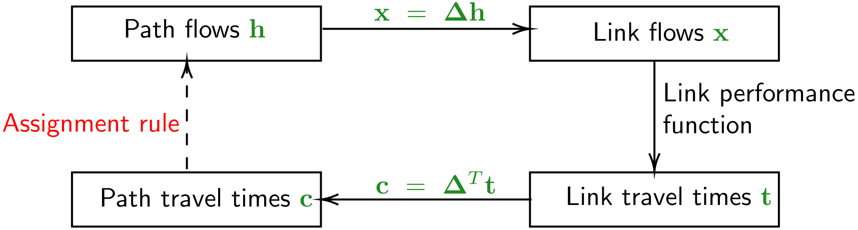
X is compact (closed & bounded) and convex.

Proof.

Since every feasible assignment $\mathbf{x} \in X$ is obtained by linear transformation of some $\mathbf{h} \in H$. □

Iterative process[1]

- We can use path flows to evaluate link flows
- We can use link flows to evaluate link travel times
- We can use link travel times to evaluate the path travel times.
- But we are still looking for a principle to use path travel times to evaluate path flows. Such principle is called assignment principle. One example of assignment principal is UE principle.



[1] BLU book Chapter 5

Mathematical modeling of UE

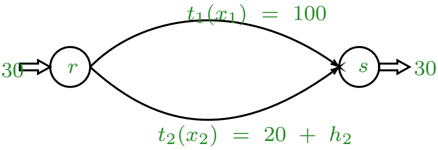
Assumptions

1. Network is strongly connected, i.e., there exists at least one path between each origin destination pair.
2. Link travel time function is positive and continuous function of its flow.

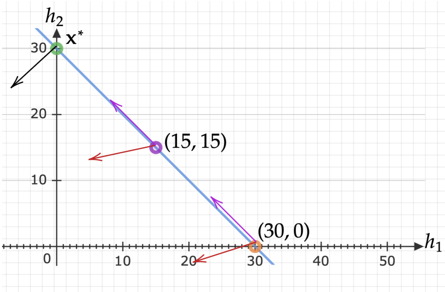
Example

Consider the following example:

UE is $\mathbf{h}^* = \begin{bmatrix} 0 \\ 30 \end{bmatrix}$ and $\mathbf{c}(\mathbf{h}^*) = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$ Consider the following cases:



- Consider $\mathbf{h} = \begin{bmatrix} 30 \\ 0 \end{bmatrix}$
  $(\mathbf{h} - \mathbf{h}^*)^T \mathbf{c}(\mathbf{h}^*) =$
  $\left(\begin{bmatrix} 30 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 30 \end{bmatrix} \right)^T \begin{bmatrix} 100 \\ 50 \end{bmatrix} = 1500$
  $\theta = \cos^{-1} \left(\frac{1500}{\sqrt{1800} \times \sqrt{12500}} \right) = 71.57°$
- Consider $\mathbf{h} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$
  $(\mathbf{h} - \mathbf{h}^*)^T \mathbf{c}(\mathbf{h}^*) =$
  $\left(\begin{bmatrix} 15 \\ 15 \end{bmatrix} - \begin{bmatrix} 0 \\ 30 \end{bmatrix} \right)^T \begin{bmatrix} 100 \\ 50 \end{bmatrix} = 750$
  $\theta = \cos^{-1} \left(\frac{750}{\sqrt{450} \times \sqrt{12500}} \right) = 71.57°$



UE as a variational inequality problem

The user equilibrium path flow vector $\mathbf{h}^* \in H$ is a solution to the following variational inequality problem VI$(H, \mathbf{c})$.

$$\boxed{\mathbf{c}(\mathbf{h}^*)^T(\mathbf{h} - \mathbf{h}^*) \geq 0, \forall \mathbf{h} \in H} \tag{13}$$

Remark. This can be interpreted as follows. $\mathbf{c}(\mathbf{h}^*)^T\mathbf{h} \geq \mathbf{c}(\mathbf{h}^*)^T\mathbf{h}^*$, i.e., any deviation from UE path flows while keeping the UE path travel times fixed cannot reduce TSTT.

Remark. Note that $(\Delta^T\mathbf{t}(\mathbf{x}^*))^T(\mathbf{h} - \mathbf{h}^*) = \mathbf{t}(\mathbf{x}^*)^T(\Delta\mathbf{h} - \Delta\mathbf{h}^*) = \mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*)$. The VI can be cast in terms of link flows, i.e.,

$$\boxed{\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X} \tag{14}$$

UE as a variational inequality problem

Theorem (BLU Theorem 6.1)

A path flow vector $\mathbf{h}^* \in H$ *is a solution to* VI$(H, \mathbf{c})$ (13) *if and only if it satisfies the UE principle.*

Proof.

This can be proved using the following contrapositive argument:
A path flow vector $\mathbf{h}^*$ *is not a solution to* VI$(H, \mathbf{c})$ (13) *if and only if it does not satisfy the UE principle.* .

$\implies$ Let us assume that $\mathbf{h}^*$ is not a solution to VI$(H, \mathbf{c})$ which means, $\exists \tilde{\mathbf{h}} \in H$ such that $\mathbf{c}(\mathbf{h}^*)^T(\tilde{\mathbf{h}} - \mathbf{h}^*) > 0 \implies \mathbf{c}(\mathbf{h}^*)^T\tilde{\mathbf{h}} > \mathbf{c}(\mathbf{h}^*)^T\mathbf{h}^*$ which shows that $\tilde{\mathbf{h}}$ is not UE since travelers can shift their paths to further reduce the overall system travel time.

$\impliedby$ Let us assume that $\mathbf{h}^*$ is not UE flow vector, which means there exists a path $\tilde{\pi} \in \Pi^{rs}$ for some $(r, s) \in Z^2$ such that $h^{\tilde{\pi}} > 0$ even though $c^{\tilde{\pi}}(\mathbf{h}^*) > \min_{\pi \in \Pi^{rs}} c^{\pi}(\mathbf{h}^*)$. Let $\pi^{'} \in \text{argmin}_{\pi \in \Pi^{rs}} c^{\pi}(\mathbf{h}^*)$. Now shift some flow $0 < \epsilon < h^{\tilde{\pi}}$ from path $\tilde{\pi}$ to path $\pi^{'}$ creating a new flow vector $\mathbf{h}^{'}$ which is a feasible flow vector. We observe that $\mathbf{c}(\mathbf{h}^*)^T(\mathbf{h}^{'} - \mathbf{h}^*) = \epsilon(c^{\pi^{'}}(\mathbf{h}^*) - c^{\tilde{\pi}}(\mathbf{h}^*)) < 0$, which shows that $\mathbf{h}^*$ is not a solution to VI$(H, \mathbf{c})$. □

UE as a fixed point problem

Let $\mathbf{h}^*$ be the equilibrium flow vector. Then,

$$\boxed{\mathbf{h}^* = \text{proj}_H(\mathbf{h} - \mathbf{c}(\mathbf{h}^*))} \tag{15}$$

In terms of link flows,

$$\boxed{\mathbf{t}^* = \text{proj}_X(\mathbf{x} - \mathbf{t}(\mathbf{x}^*))} \tag{16}$$

Theorem

If assumptions on slide 10 holds, then there exists at least one UE solution.

Proof.

Use Brouwer's fixed point theorem taught in the previous lecture. The existence of solution can also be proved using Weierstrass' Theorem (refer to lecture on *Mathematical Preliminaries*) for the Beckmann's formulation presented on slide 19. □

UE as a fixed point problem

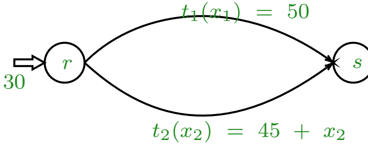
Another way to pose UE as a fixed point problem is composite mapping. The UE assignment rule is a function $\mathcal{A}^{\mathsf{UE}}$ that returns the equilibrium path flows $\mathbf{h}^*$ given the equilibrium path travel times $\mathbf{c}$ which is a continuous function of equilibrium path flows $\mathbf{h}^*$. The overall problem can be written as:

$$\boxed{\mathbf{h}^* \in \mathcal{A}^{\mathsf{UE}}(\mathbf{c}(\mathbf{h}^*))} \qquad (17)$$

Remark.

1. $\mathbf{c}(\mathbf{h}) = \mathbf{\Delta}^T \mathbf{t}(\Delta \mathbf{h})$
2. $\mathcal{A}^{\mathsf{UE}}(\mathbf{c}) = \{\mathbf{h} \in H \mid h^\pi > 0 \iff c^\pi = \min_{\pi' \in \Pi^{rs}} \{c^{\pi'}\}, \forall \pi \in \Pi^{rs}, \forall (r,s) \in Z^2\}$ Notice that $f(\mathbf{h}) = \mathcal{A}^{\mathsf{UE}}(\mathbf{c}(\mathbf{h}^*))$ is a set-valued map from H to 2^H. To make it clear, check the following example:

$$\mathcal{A}^{\mathsf{UE}}(c_1, c_2) = \begin{cases} \{[30, 0]\} & c_1 < c_2 \\ \{[0, 30]\} & c_1 > c_2 \\ \{[h, 30-h] \mid h \in [0, 30]\} & c_1 > c_2 \end{cases}$$



Theorem

Under assumptions on slide 10, there is at least one solution to the fixed point problem $\mathbf{h}^* \in \mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}^*))$.

Proof.

We use Kakutani's fixed point theorem to prove this.

- H is non-empty, closed, bounded, and convex (already proved).
- $\mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}))$ is non-empty since given $\mathbf{h} \in H$ there will be at least one path with minimum travel time.
- $\mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}))$ is convex. Take any feasible path flow vectors $\mathbf{h}^1, \mathbf{h}^2 \in H$. The convex combination $\mathbf{h} = \lambda * h^1 + (1-\lambda)\mathbf{h}^2 \in H$ since H is a convex set. For feasible $\mathbf{h} \in H$, the mapping $\mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}))$ will select the shortest travel time paths for assigning flow.
- $\mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}))$ has a closed graph property. We need to show that a sequence $\{(\mathbf{h}_k, \hat{\mathbf{h}}_k)\}_k$, where $\hat{\mathbf{h}}_k \in \mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}_k))$, will converge to $(\mathbf{h}, \hat{\mathbf{h}})$, where $\hat{\mathbf{h}} \in \mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}))$. Note that $c(\mathbf{h})$ is a continuous function in $\mathbf{h}$ if link travel time functions are continuous. Let's assume that $\{(\mathbf{h}_k, \hat{\mathbf{h}}_k)\}_k$ converges to $(\mathbf{h}, \tilde{\mathbf{h}})$, where $\tilde{\mathbf{h}} \notin \mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}))$. This means

  $\exists \pi \in \Pi^{rs}$ for some $(r,s) \in Z^2$ such that $c^{\pi}(\mathbf{h}) > \min_{\pi' \in \Pi^{rs}} \{c^{\pi'}(\mathbf{h})\} + \epsilon$ for some $\epsilon > 0$. But since $\mathbf{c}(\mathbf{h}_k)$ converges to $\mathbf{c}(\mathbf{h})$, there exists a large enough K such for $k \geq K$, $c^{\pi}(\mathbf{h}_k) > \min_{\pi' \in \Pi^{rs}} \{c^{\pi'}(\mathbf{h}_k)\} + \frac{\epsilon}{2}$. This means for $k \geq K$, $\hat{h}_k^{\pi} = 0$ but then $\hat{\mathbf{h}}_k \notin \mathcal{A}^{\text{UE}}(\mathbf{c}(\mathbf{h}_k))$, a contradiction.

UE as a nonlinear complementarity problem

Let k^{rs} be the cost of the path with shortest travel time between O-D pair $(r, s) \in Z^2$. Further let $\mathbf{k} = \{k^{rs}\}_{\pi \in \Pi^{rs}, (r,s) \in Z^2}$ UE can be formulated as NCP as below:

$$(\mathbf{c}(\mathbf{h}^*) - \mathbf{k})^T \mathbf{h}^* = 0 \quad (18)$$

$$(\mathbf{c}(\mathbf{h}^*) - \mathbf{k}) \geq 0 \quad (19)$$

$$\mathbf{h} \in H \quad (20)$$

UE as a convex optimization problem

We studied that for a convex optimization problem $\min_{\mathbf{x}}\{f(\mathbf{x}) \mid \mathbf{x} \in X\}$ with differentiable f, $\mathbf{x}^* \in X$ is optimal if and only if

$$\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X \tag{21}$$

We also stated that UE link flow vector $\mathbf{x}^* \in X$ is a solution to the following VI$(X, \mathbf{t})$:

$$\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X \tag{22}$$

Comparing (21) and (22), we see that

$$t_{ij}(x_{ij}) = \frac{\partial F(x_{ij})}{\partial x_{ij}} \implies \boxed{F(x_{ij}) = \int_0^{x_{ij}} t_{ij}(x)dx}$$

$F(x_{ij})$ is also known as Beckmann's function. Therefore, UE can be formulated as a following optimization problem:

$$Z^{UE} = \underset{\mathbf{x}}{\text{minimize}} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx \tag{23a}$$

$$\text{subject to} \quad \mathbf{x} \in X \tag{23b}$$

Beckmann's formulation of UE

$$Z^{UE} = \underset{\mathbf{x},\mathbf{h}}{\text{minimize}} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx \tag{24a}$$

$$\text{subject to} \quad \sum_{\pi\in\Pi^{rs}} h^\pi = d^{rs}, \forall (r,s) \in Z^2 \tag{24b}$$

$$h^\pi \geq 0, \forall \pi \in \Pi \tag{24c}$$

$$x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^\pi h^\pi, \forall (i,j) \in A \tag{24d}$$

Remark. Beckmann's function does not have any physical, behavioral, or economic interpretation.

Beckmann's formulation in terms of link flows

$$Z^{UE} = \min_{\mathbf{x}} \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx \tag{25a}$$

$$\text{s.t.} \quad x_{ij}^{rs} \geq 0, \forall (i,j) \in A \tag{25b}$$

$$\sum_{j\in FS(i)} x_{ij}^{rs} - \sum_{j\in BS(i)} x_{ji}^{rs} = \begin{cases} d^{rs}, & \text{if } i = r \\ -d^{rs}, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases}, \forall i \in N, \forall (r,s) \in Z^2 \tag{25c}$$

$$x_{ij} = \sum_{(r,s)\in Z^2} x_{ij}^{rs}, \forall (i,j) \in A \tag{25d}$$

Beckmann's formulation in terms of path flows

$$Z^{UE} = \underset{\mathbf{h}}{\text{minimize}} \quad \sum_{(i,j)\in A} \int_0^{\sum_{\pi\in\Pi} \delta_{ij}^{\pi} h^{\pi}} t_{ij}(x)dx \tag{26a}$$

$$\text{subject to} \quad \sum_{\pi\in\Pi^{rs}} h^{\pi} = d^{rs}, \forall (r,s) \in Z^2 \tag{26b}$$

$$h^{\pi} \geq 0, \forall \pi \in \Pi \tag{26c}$$

Theorem

The optimality conditions of Beckmann's optimization problem (26) *corresponds to UE conditions .*

Proof.

Let us associate dual variables $\{k^{rs}\}_{(r,s)\in Z^2}$ and $\{\lambda^{\pi}\}_{\pi\in\Pi}$ to constraints (26b) and (26c) respectively. The Lagrangian function can be written as:

$$L(\mathbf{h}, \mathbf{k}, \lambda)$$

$$= \sum_{(i,j)\in A} \int_0^{\sum_{\pi\in\Pi} \delta_{ij}^{\pi} h^{\pi}} t_{ij}(x)dx + \sum_{(r,s)\in Z^2} k^{rs}(d^{rs} - \sum_{\pi\in\Pi^{rs}} h^{\pi}) - \sum_{\pi} \lambda^{\pi} h^{\pi}$$

The KKT conditions of (26) are given below:

1. *Primal conditions*

$$\sum_{\pi\in\Pi^{rs}} h^{\pi} = d^{rs}, \forall (r,s) \in Z^2 \tag{27}$$

$$-h^{\pi} \leq 0, \forall \pi \in \Pi \tag{28}$$

□

Proof (contd.)

2. *Dual conditions*

$$\lambda^\pi \geq 0, \forall \pi \in \Pi \tag{29}$$

3. *Complementary slackness*

$$\lambda^\pi h^\pi = 0, \forall \pi \in \Pi \tag{30}$$

4. *Gradient of Lagrangian vanishes at* $\mathbf{h}$

$$\sum_{(i,j)\in A} \delta_{ij}^\pi t_{ij}(x_{ij}) - k^{rs} - \lambda^\pi = 0 \tag{31}$$

(31) implies that $\lambda^\pi = c^\pi - k^{rs}, \forall \pi \in \Pi$. Further, using (30), we have

$$\boxed{h^\pi(c^\pi - k^{rs}) = 0, \forall \pi \in \Pi}$$

We observe that

1. If $h^\pi > 0 \implies c^\pi = k^{rs}$
2. If $c^\pi > k^{rs} \implies h^\pi = 0$

This is exactly UE. Note that k^{rs} is the cost of shortest path between $(r, s) \in Z^2$, □

You might have to use the following rule for taking the derivative in (31).

Leibnitz's Rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x)) \cdot \frac{d}{dx}b(x) - f(x,a(x)) \cdot \frac{d}{dx}a(x) + \left(\int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt\right)$$

Uniqueness of UE

Assumption: Network is strongly connected and link travel time function is a non-negative, continuous, and strictly increasing function of its flow, i.e.,

- $t_{ij}(x_{ij}) \geq 0, \forall (i,j)$
- $\frac{\partial t_{ij}(x_{ij})}{\partial x_{pq}} = 0, \forall (i,j) \neq (p,q)$
- $\frac{\partial t_{ij}(x_{ij})}{\partial x_{ij}} > 0, \forall (i,j)$

Theorem

Under above assumptions, Beckmann's formulation will produce unique link flows.

Proof.

Since $t_{ij}(x_{ij})$ is a strictly increasing function of x_{ij}, $\int_0^{x_{ij}} t_{ij}(x)dx$ is a strict convex function (result proved in *Mathematical preliminaries* lecture).The objective function sum of strictly convex functions, therefore it is strictly convex. Further, the feasible region of the optimization problem is also convex set. Therefore, (25) produces a unique minimizer. This result can also be proved through VI. □

Uniqueness of UE

Although Beckmann's formulation will produce unique solution in the link flow space but it may not produce unique solution in the path flow space. This is because the Beckmann's function in terms of path flows may not be strictly convex. Multiple path flows can be produced for a given link flow solution.

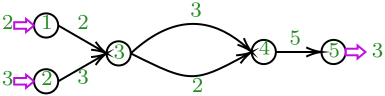
Uniqueness of UE

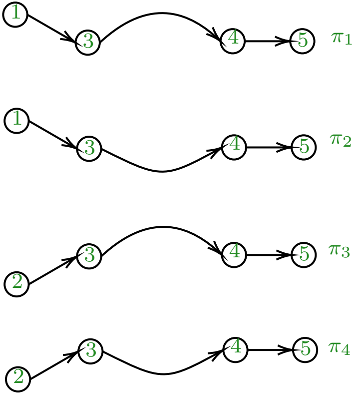
For example[2], if $x_{13} = 2, x_{23} = 3, x_{34}^{\text{upper}} = 3, x_{34}^{\text{lower}} = 2, x_{45} = 5$, we can have the following path flows.

1. $h^{\pi_1} = 0, h^{\pi_2} = 2, h^{\pi_3} = 3, h^{\pi_4} = 0$
2. $h^{\pi_1} = 2, h^{\pi_2} = 0, h^{\pi_3} = 1, h^{\pi_4} = 2$

In fact, there are infinite combination of path flow vectors for the given link flow values.

$h^{\pi_1} = 2\alpha, h^{\pi_2} = 2(1-\alpha), h^{\pi_3} = 3 - 2\alpha, h^{\pi_4} = 2\alpha, \forall \alpha \in [0, 1]$





[2]Example taken from Sheffi Chapter 3

System optimal traffic assignment

In SO assignment, we minimize TSTT.

$$Z^{SO} = \underset{\mathbf{h},\mathbf{x}}{\text{minimize}} \quad \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}) \tag{32a}$$

$$\text{subject to} \quad \sum_{\pi\in\Pi^{rs}} h^\pi = d^{rs}, \forall (r,s)\in Z^2 \tag{32b}$$

$$h^\pi \geq 0, \forall \pi \in \Pi \tag{32c}$$

$$x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^\pi h^\pi, \forall (i,j)\in A \tag{32d}$$

Proposition

Beckmann's formulation of UE (24) *will produce the SO assignment flows if link travel time functions are replaced by marginal travel time functions* $\tilde{t}_{ij}(x) = t_{ij}(x) + x_{ij}\frac{dt_{ij}(x)}{dx}$ *in the objective function* (24a), *where* $x_{ij}\frac{dt_{ij}(x)}{dx}$ *is the externality caused by additional traveler.*

Proof.

Since $\sum_{(i,j)\in A}\int_0^{x_{ij}} \tilde{t}_{ij}(x) = \sum_{(i,j)\in A} x_{ij}t_{ij}(x_{ij})$ □

Remark. This means by imposing the toll equal to $x^{SO}\frac{dt_{ij}(x^{SO})}{dx}$ on each link, one can achieve system optimal traffic state.

Suggested reading

- Sheffi Chapter 3
- Patriksson Chapter 2 and 3
- BLU book Chapter 6
- Dafermos, Stella. "Traffic equilibrium and variational inequalities." Transportation science 14.1 (1980): 42-54.

Beckmann, McGuire, and Winsten (1955)

U. S. AIR FORCE
PROJECT RAND
RESEARCH MEMORANDUM

STUDIES IN THE ECONOMICS OF TRANSPORTATION *

Martin Beckmann
C. B. McGuire
Christopher B. Winsten

RM-1488

12 May 1955

will give us a proof that there exist solutions to our system. Consider the function

$$(3.13)\qquad H(\ldots . x_{ij,k}, \ldots) = \sum_{i,k} \int_0^{x_{i,k}} g_{i,k}(x)\,dx - \frac{1}{2} \sum_{ij} \int_0^{x_{ij}} h_{ij}(x)\,dx.$$

(Economists are warned that this is not to be interpreted as consumers' surplus! The term on the right has as its kernel the average, rather than the incremental cost to users collectively.) Here the factor $\frac{1}{2}$ comes in because we wish to sum over every road but once, while each road is denoted by two pairs of indices, ij and ji. Differentiate with respect to $x_{ij,k}$ after substituting for $x_{i,k}$ and x_{ij} from (3.4) and (3.2). The

Smith (1979)

THE EXISTENCE, UNIQUENESS AND STABILITY OF TRAFFIC EQUILIBRIA

M. J. SMITH
Department of Mathematics, University of York, Heslington, York YO1 5DD, England

(*Received 22 May 1978; in revised form 1 February 1979*)

Let **H** satisfy the condition (8) for a Wardrop equilibrium, and let **C(H)** be the route-costs determined by **H**. *Regard these route-costs as fixed.* Because **H** satisfies (8) and so only cheapest routes are used, total cost cannot be reduced by any change or changes of route (route-costs, remember, are fixed). Therefore any other route-flow **F** in Δ has total cost at least as great as the route-flow **H** which uses only cheapest routes; it then follows from (4), the definition of total cost, that

$$\mathbf{C}(\mathbf{H}) \cdot \mathbf{F} \geqslant \mathbf{C}(\mathbf{H}) \cdot \mathbf{H} \text{ for all } \mathbf{F} \in \Delta \tag{9}$$

Conversely, suppose that (8) is not satisfied. Then there is $(x, y) \in N \times N$ and routes R_r, $R_s \in \mathcal{R}(x, y)$ such that

$$H_r > 0 \text{ and } C_r(\mathbf{H}) > C_s(\mathbf{H}).$$

Moving the flow H_r along R_r to the cheaper route R_s will reduce total cost by $C_r(\mathbf{H})H_r - C_s(\mathbf{H})H_r > 0$. Thus, if the resulting route-flow is **F**,

$$\mathbf{C}(\mathbf{H}) \cdot \mathbf{F} < \mathbf{C}(\mathbf{H}) \cdot \mathbf{H} \text{ and } \mathbf{F} \in \Delta.$$

In which case (9) is not satisfied.

We have shown here that if (8) is satisfied then (9) is satisfied. We have also shown that if (8) is not satisfied then (9) is not satisfied. Hence (8) and (9) are equivalent.

It is clear that (9) is in turn equivalent to each of the two following conditions:

$$[-\mathbf{C}(\mathbf{H})] \cdot (\mathbf{F} - \mathbf{H}) \leq 0 \quad \text{for all } \mathbf{F} \in \Delta. \tag{10}$$