Minimum spanning tree

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Definitions

Definition (Spanning Tree). A spanning tree of undirected network G is connected acyclic subgraph that spans all the nodes.

Definition (Minimum spanning tree (MST) problem). Given an undirected network $G(N, A)$ and costs $c : E \mapsto \mathbb{R}$, determine a spanning tree T with minimum cost $\sum_{(i,j)\in T} c_{ij}$.

Remark. We consider the undirected network for spanning tree. In case of directed network, the problem (much more difficult problem) is known as rooted aborescence. For a node r, r-aborescence is a spanning tree directed away from r. There is only one directed path from r to every other node.

Remark. For maximum spanning tree, just multiply each cost with by -1 and compute the MST.

Applications

1. Creating a minimal transit network,
2. Connecting different spatial areas with electricity connection,
3. Clustering (based on Kruskal's algorithm),

   and so on...

LP formulation

Primal

$$\min_{\mathbf{x}} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{(i,j)\in A} x_{ij} = n-1$$

$$\sum_{(i,j)\in A(S)} x_{ij} \leq |S|-1, \forall S \subset N$$

$$x_{ij} \geq 0, \forall (i,j) \in A$$

Dual

$$\max_{\lambda,\mu_S} (n-1)\lambda + \sum_{S\subset N} (1-|S|)\mu_S$$

$$\text{s.t. } \lambda - \sum_{S:(i,j)\in A(S)} \mu_S \leq c_{ij}, \forall (i,j) \in A$$

$$\mu_S \geq 0, \forall S \subset N$$

λ free

Generic MST algorithm

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1: Input: G, c
2: (Initialization) A = φ
3: while A does not form a tree do
4:     find a "safe" edge (i, j) for A
5:     A = A ∪ {(i, j)}
6: end while
7: return A
```

We maintain the following loop invariant.

Prior to each iteration, A is a subset of some MST

Definition (Safe edge).: Any edge added to A satisfying the above loop invariant is called a safe edge.

1. *Initialization*: After line 2, the loop invariant is trivially satisfied.
2. *Maintenance*: The lines 3-5 maintain the loop invariant by only adding "safe" edges.
3. *Termination*: All the edges added to A were part of MST, so after termination, the loop invariant must hold.

Q. How to find safe edge?

A few more definitions

Definition (Cut). Any partition $(S, N \backslash S)$ is a cut. We say that an edge (i, j) crosses the cut $(S, N \backslash S)$ if one of the endpoints is in S and other endpoint in $N \backslash S$.

Definition (Tree/Non-tree edges). Edges in a given spanning tree are tree edges, otherwise they are non-tree edges.

Important observations

1. For every non-tree edge (i, j), a spanning tree T has a unique path connecting i and j. Adding edge (i, j) to T will create a cycle.
2. Removing any tree edge from a spanning tree will create a cut.

Optimality conditions

Theorem (Cut optimality conditions)

A spanning tree T^* is a minimum spanning tree (MST) if and only if it satisfies the following optimality conditions: For every tree edge $(i, j) \in T^*$, $c_{ij} \leq c_{kl}$ for every edge (k, l) contained in the cut formed by removing the edge (i, j) from T^*.

Proof.

$\Longrightarrow$ Assume that T^* is MST. Further, assume that the cut optimality conditions are not satisfied, i.e., $\exists$ a tree edge $(i, j) \in T^*$ removing which creates a cut and $\exists$ an edge (k, l) (in original graph) crossing the cut which has cost c_{kl} strictly less than c_{ij}. Then, replacing the edge (i, j) by (k, l) will produce another tree $T^{'}$ whose overall cost strictly less than T^*, which is a contradiction that T^* is MST.

$\Longleftarrow$ We need to show that if any tree T^* satisfies the cut optimality conditions, then it must be MST. Suppose $T^{'}$ is a MST such that $T^{'} \neq T^*$. Then, there must exist an edge (i, j) in T^* which is not present in $T^{'}$. Removing the edge (i, j) from T^* creates a cut $(S, N \backslash S)$. Note that if we add (i, j) to $T^{'}$, then it will create a cycle that must contain another edge (k, l) crossing the cut. Since T^* satisfies the cut optimality conditions $c_{ij} \leq c_{kl}$ and since $T^{'}$ is MST, $c_{ij} \geq c_{kl}$ implies that $c_{ij} = c_{kl}$. Now, we replace (i, j) by (k, l) in T^*, we produce a different spanning tree which has one or more edges common with $T^{'}$. Repeating this argument, we can transform T^* into MST $T^{'}$. This shows that T^* is also a MST. $\square$

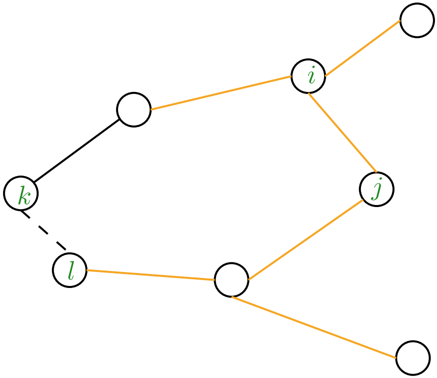


Figure: Replacing (i, j) by (k, l)

Remark. The cut optimality conditions imply that every edge in a MST is a minimum cost edge across the cut that is defined by removing it from the tree.

Optimality conditions

Theorem

Let F is a subset of edges in some MST and let S be a set of nodes in some component of F. Suppose (i,j) is a minimum cost edge in the cut $(S, N \backslash S)$. Then some MST contains all the edges of in F as well as edge (i,j).

Proof.

Let $F \subseteq T^*$ (MST). If $(i,j) \in T^*$, we are done. Therefore, suppose $(i,j) \notin T^*$. Then, adding (i,j) to T^* creates a cycle and therefore, $\exists (k,l) \neq (i,j) \in (S, N \backslash S)$. By assumption, $c_{ij} \leq c_{kl}$ and also T^* must satisfy the cut optimality conditions which says $c_{ij} \geq c_{kl}$. So replacing (k,l) by (i,j) will produce another MST that contains F as well as (i,j). □

Optimality conditions

Theorem (Path optimality conditions)

A spanning tree T^* is a MST if and only if satisfies the following path optimality conditions: For every non-tree edge (k,l) of G, $c_{ij} \leq c_{kl}$ for every edge (i,j) contained in the path in T^* connecting nodes k and l.

Proof.

$\Longrightarrow$ Suppose T^* is a MST and $\exists$ a non-tree edge (k,l) and a tree edge (i,j) contained in the path connecting k and l such that $c_{ij} > c_{kl}$. In that case, we can remove (i,j) and add (k,l) creating another tree $T^{'}$ with cost $c(T^{'}) < c(T^*)$, contradicting the assumption that T^* is a MST.

$\Longleftarrow$ We'll show that T^* satisfying the path optimality conditions also satisfy the cut optimality conditions, implying that T^* is a MST using previous theorem. Let $(i,j) \in T^*$ and let S and $\bar{S}$ be the set of connected nodes produced by removing edge (i,j) from T^*. Suppose $i \in S$ and $j \in \bar{S}$. Consider any edge $(k,l) \in (S,\bar{S})$. Since T^* contains a unique path joining nodes k and l and since (i,j) is the only edge connecting a node in S and a node in $\bar{S}$, edge (i,j) must belong to this path. The path optimality conditions implies that $c_{ij} \leq c_{kl}$; since this condition must be valid for every nontree edge (k,l) in the cut $(S,\bar{S})$ formed by removing any tree edge (i,j), T^* satisfy the cut optimality conditions and so it must be MST. □

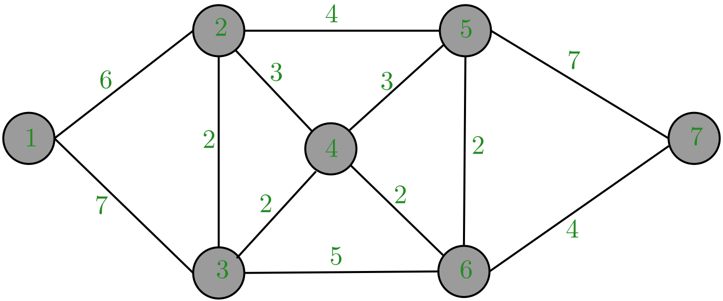
Kruskal's algorithm

Input: G, c
(*Initialization*) $A \leftarrow \phi$ ▷ Links in MST
for each $i \in N$ **do**
    MakeSet(i)
end for
Let A be the set of links sorted in increasing order by their costs c.
for each $(i, j) \in A$ **do**
    if FindSet(i) $\neq$ FindSet(j) **then**
        $A = A \cup \{(i, j)\}$
        Union(i, j)
    end if
end for
return A

Theorem

Kruskal's algorithm can be implemented in $O(m \log n)$ time.

Example



Prim's algorithm

- Based on the cut-optimality condition.
- Maintains for every node $d(i)$ and $pred(i)$ representing minimum cost of any edge connecting i to another node in tree and predecessor respectively.
- Also maintains a heap Q of all nodes not in the tree yet.
  - Heap is a data structure having a collection of objects with unique key.
  - We can perform operations such as $\textsc{CreateHeap}()$, $\textsc{Insert}(i, Q)$, $\textsc{DecreaseKey}(Q, j, c_{ij})$, etc.
  - Check out `heapq` in Python.

Prim's algorithm

1: **Input**: G, c, s ▷ Graph, costs, source
2: $d(i) \leftarrow \infty, \forall i \in N\{s\}; d(s) \leftarrow 0$
3: $pred(i) \leftarrow \texttt{NA}, \forall i \in N\backslash\{s\}$
4: $Q \leftarrow$ CreateHeap() ▷ Creates a heap Q
5: **for** each $i \in N$ **do**
6: Insert(Q, i) ▷ Inserts node i into heap Q
7: **end for**
8: **while** Q **do**
9: $i \leftarrow$FindMin(Q)
10: Delete(Q, i)
11: **for** $j \in \delta(i)$ **do**
12: **if** $j \in Q$ **and** $d(j) > c_{ij}$ **then**
13: $d(j) \leftarrow c_{ij}$
14: $pred(j) \leftarrow i$
15: DecreaseKey(Q, j, c_{ij}) ▷ Reduces the key of j in Q to c_{ij}
16: **end if**
17: **end for**
18: **end while**
19: **return** $A = \{(pred(i), i) : i \in N\backslash\{s\}\}$

Theorem

Above algorithm runs in $O(m \log n)$ time.

Lines4-6: $O(n)$ time; while runs: $O(n)$ times; 9-10: $O(logn)$ time; Line 15: $O(logn)$ time. For loop runs: $O(m)$ times;

Sollin's algorithm

- Sollin's algorithm combines ideas from both Kruskal's and Prim's algorithm.
- It maintains a set of forests (like Kruskal's) but only selects the edge with minimum cost (like Prim's).
- Running time $O(m \log n)$.

Suggested reading

- AMO Chapter 13

Thank you!