Minimum cost flow problem

Pramesh Kumar

IIT Delhi

April 4, 2024

Introduction

- Shortest path problems model link cost but not link capacity. On the other hand, the max flow problem models link capacity but not link cost.
- Mincost flow problem (MCFP) models both link costs as well as link capacity.
- It is fundamental problem with numerous applications such as production planning, scheduling, transportation of goods, etc.

Definition (Minimum cost flow problem). Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, lower and upper bounds (capacity) on the flow on links $l : A \mapsto \mathbb{R}$ and $u : A \mapsto \mathbb{R}$ resp., and supply/demand at each node $b : N \mapsto \mathbb{R}$, find the least cost shipment of a commodity.

Assumptions

- All data (costs, capacities, supply/demand) are integral.
- The network is directed.
- Supply/demand balance, i.e., $\sum_{i \in N} b(i) = 0$ and MFCP has a feasible solution.[1]
- All costs are non-negative.

[1] One can find a feasible solution to MCFP by solving a max flow problem on a modified network with a "super source" connecting each supply node using a link with capacity $b(i)$ and a "super sink" connecting each demand node using a link with capacity $b(i)$. If the max flow saturates all the source links, then that flow is a feasible solution to MCFP.

LP formulation

Primal

$$\min_{\mathbf{x}} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j\in FS(i)} x_{ij} - \sum_{j\in BS(i)} x_{ji} = b(i), \forall i \in N$$

$$0 \le x_{ij} \le u_{ij}, \forall (i,j) \in A$$

where, if $b(i) > 0, b(i) < 0$, and $b(i) = 0$, then i is called supply node, demand node, and transshipment node respectively.

Dual

$$\max_{\mathbf{d},\alpha} \sum_{i\in N} b(i)\pi_i - \sum_{(i,j)\in A} \alpha_{ij}u_{ij}$$

$$\text{s.t. } \pi_i - \pi_j - \alpha_{ij} \le c_{ij}, \forall (i,j) \in A$$

$$\alpha_{ij} \ge 0$$

$$\pi_i \text{ free }, \forall i \in N$$

$$= \max_{\pi} \sum_{i\in N} b(i)\pi_i - \sum_{(i,j)\in A} \max\{-c^{\pi}_{ij}, 0\}u_{ij}$$

where, $\boxed{c^{\pi}_{ij} = c_{ij} - \pi_i + \pi_j}$ is called reduced cost of link $(i,j) \in A$

Residual network

Residual network corresponding to flow $\mathbf{x}$ is created as below:

- Replace each link (i, j) by two links (i, j) and (j, i).
- Put c_{ij} cost on link (i, j) and $-c_{ij}$ cost on link (j, i).
- Put $r_{ij} = u_{ij} - x_{ij}$ as the residual capacity on link (i, j) and $r_{ji} = x_{ij}$ as the residual capacity on link (j, i).
- Remove links with zero residual capacity.

Negative cycle optimality conditions

Theorem

A feasible solution $\mathbf{x}^*$ *is an optimal solution if and only if the residual network* $G(\mathbf{x}^*)$ *contains no negative cost (directed) cycles.*

Reduced costs

Given node potentials (or dual variables corresponding to conservation constraints) $\pi(i)$, $\boxed{c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)}$ is called the reduced cost of link $(i,j) \in A$.

Economic interpretation: $-\pi(i)$: cost of obtaining a unit of this commodity at node i. $c_{ij} - \pi(i)$: the cost of obtaining unit commodity to node j from node i.

Lemma

1. *For a directed path P from node k to node l,* $\sum_{(i,j)\in P} c_{ij}^{\pi} = \sum_{(i,j)\in P} c_{ij} - \pi(k) + \pi(l)$.
2. *For a directed cycle W,* $\sum_{(i,j)\in W} c_{ij}^{\pi} = \sum_{(i,j)\in W} c_{ij}$

Proof.

1. *Let* $P = \{k = i_1, ..., i_h = l\}$ *be a directed path. Then,* $\sum_{(i,j)\in P} c_{ij}^{\pi}$ $= c_{i_1 i_2}^{\pi} + \cdots + c_{i_{h-1} i_h}^{\pi} = (c_{i_1 i_2} - \pi(i_1) + \pi(i_2)) + \cdots + (c_{i_{h-1} i_h} - \pi(i_{h-1}) + \pi(i_h)) = \sum_{(i,j)\in P} c_{ij} - \pi(k) + \pi(l)$.
2. *Trivial.*

Remark. 2 implies that if W is a negative cost cycle wrt costs c_{ij}, then it is also a negative cycle wrt costs c_{ij}^{π}.

Reduced costs optimality conditions

Theorem

A feasible solution $\mathbf{x}^*$ *is an optimal solution if and only if there exists node potentials* π *which satisfy* $c_{ij}^{\pi} \geq 0, \forall (i,j) \in G(\mathbf{x}^*)$.

Proof.

$\Longleftarrow$ Assume that for a feasible solution $\mathbf{x}^*$, $c_{ij}^{\pi} \geq 0, \forall (i,j) \in G(\mathbf{x}^*)$. Then, we know that $\sum_{(i,j)\in W} c_{ij}^{\pi} \geq 0$ for every directed cycle W in $G(\mathbf{x}^*)$ (previous lemma). Then, there does not exists any cycle with negative cost. Using negative cycle optimality conditions, we know that $\mathbf{x}^*$ is optimal.

$\Longrightarrow$ Assume that $\mathbf{x}^*$ is optimal, then using negative cycle optimality conditions, we know that there are no negative cost directed cycles in $G(\mathbf{x}^*)$. Then, find the shortest path from node 1 (w.l.o.g.) to all other nodes. Since, there are no no negative cost directed cycles, we can find shortest path labels $d(i)$ for all nodes satisfying $d(j) \leq d(i) + c_{ij}, \forall (i,j) \in G(\mathbf{x}^*)$. Define $\pi(i) = -d(i), \forall i \in N$. Clearly, $c_{ij} - (-d(i)) + (-d(j)) \geq 0$.

□

Remark. $c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j) \geq 0$ means that cost of obtaining the commodity at node j is no more than the cost of the commodity if we obtain at node i and incur the transportation cost in sending it from node i to node j.

Complementary slackness optimality conditions

Theorem

A feasible solution $\mathbf{x}^*$ is an optimal solution if and only if there exists node potentials π that (together with $\mathbf{x}^*$) satisfy the following complementary slackness optimality conditions:

- *If $c^{\pi}_{ij} > 0$, then $x^*_{ij} = 0$.*
- *If $0 < x^*_{ij} < u_{ij}$, then $c^{\pi}_{ij} = 0$.*
- *If $c^{\pi}_{ij} < 0$, then $x^*_{ij} = u_{ij}$.*

Proof.

We'll show that if $\mathbf{x}^*$ and π satisfy the reduced cost optimality conditions, then they also satisfy complementary conditions.

- If $c^{\pi}_{ij} > 0$, then residual network cannot contain link (j, i) because $c^{\pi}_{ji} = -c^{\pi}_{ij} < 0$ contradicting the reduced cost optimality conditions. Therefore, $x^*_{ij} = 0$.
- If $0 < x^*_{ij} < u_{ij}$, then the residual network contains both (i, j) and (j, i). Further, reduced cost optimality conditions state that $c^{\pi}_{ij} \geq 0$ as well as $c^{\pi}_{ji} \geq 0$. But we know that $c^{\pi}_{ij} = -c^{\pi}_{ji}$. Therefore, $c^{\pi}_{ij} = 0$.
- If $c^{\pi}_{ij} < 0$, then residual network cannot contain link (i, j) since its reduced cost violates the reduced cost optimality conditions. Therefore, $x^*_{ij} = u_{ij}$.

□

Evaluating optimal node potentials given optimal flows

- Construct $G(\mathbf{x}^*)$
- Solve the shortest path from node 1 (pick arbitrarily) to all other nodes and compute distance labels $d(i)$ (which are well defined since no negative cycle exists at optimality).
- Assign $\pi(i) = -d(i)$

Remark. Above node potentials are optimal because
$c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j) = c_{ij} - (-d(i)) + (-d(j)) \geq 0 \implies d(j) \leq d(i) + c_{ij}$
which are shortest path optimality conditions.

Evaluating optimal flows given optimal node potentials

- Compute reduced cost c_{ij}^{π} of each link $(i, j) \in A$.
- If $c_{ij}^{\pi} > 0$, then assign $x_{ij}^{*} = 0$. Remove (i, j) from the network.
- If $c_{ij}^{\pi} < 0$, then assign $x_{ij}^{*} = u_{ij}$. Remove (i, j) from the network. Reduce $b(i)$ by u_{ij} and increase $b(j)$ by u_{ij}
- The network $G^{'}(N, A^{'})$ with modified supply/demand $b^{'}(i)$ at nodes.
- Add new links from "super source" to supply nodes (with capacity $b^{'}(i)$) and demand nodes to "super sink" (with capacity $-b^{'}(i)$).
- Solve the max flow problem from super source to super sink. Assign x_{ij}^{*} equal to the optimal solution of max flow problem.

Remark. Above node potentials are optimal because $c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j) = c_{ij} - (-d(i)) + (-d(j)) \geq 0 \implies d(j) \leq d(i) + c_{ij}$ which are shortest path optimality conditions.

Cycle-canceling algorithm

```
1: procedure CycleCanceling(G, c, u, b)
2:     find a feasible flow x^2 in the network.
3:     while G(x) contains a negative cycle do
4:         find a negative cycle W^3.
5:         δ = min{r_ij : (i, j) ∈ W}
6:         augment δ units of flow along W
7:         update G(x)
8:     end while
9: end procedure
```

- The upper bound on the initial cost of flow is mCU.
- The lower bound on the optimal cost of flow is $-mCU$.
- Each iteration of above algorithm changes the objective value by $\left(\sum_{(i,j)\in A} c_{ij}\right)\delta < 0$.
- Finding the cycle takes $O(mn)$ time using label correcting algorithm.
- Since data is integral, total time in running the algorithm is $O(mn \times 2mCU) = O(m^2nCU)$.

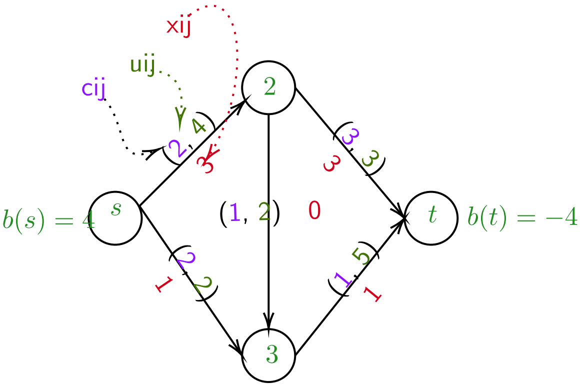
[2] possibly by solving max flow problem on a modified network

[3] possibly using label correcting algorithm

Theorem

If all link capacities and supplies/demands of nodes are integer, the minimum cost flow problem has always integer minimum cost flow.

Example



Final remarks

- We did not study many other algorithms to solve this problem. I suggest that you study the following from AMO book.
  - Successive shortest path algorithm
  - Out-of-kilter algorithm
  - Primal-dual algorithm
  - Lagrangian relaxation-based algorithm
  - Network simplex algorithm
- The algorithms we studied had pseudo-polynomial complexity. The scaling algorithms have polynomial complexity.
  - Minimum cost scaling algorithm
  - Cost scaling algorithm
  - Double scaling algorithm

Suggested reading

- AMO Chapter 9 and 10

Thank you!