

# **Modeling linear programming problems**

Pramesh Kumar

IIT Delhi

August 5, 2024

# Outline

Motivation

Optimization Framework

Reformulations

Examples

# What is optimization?

(Merriam-Webster Dictionary) An act, process, or methodology of making something (such as a design, system, or decision) as fully perfect, functional, or effective as possible.

## Example

(Maximum Area Problem)

You have 80 meters of wire and want to enclose a rectangle as large as possible (in area). How should you do it?

Note: This is a non-linear optimization problem!

# Outline

Motivation

Optimization Framework

Reformulations

Examples

# Common Framework

## Components of an optimization problem

- ▶ Decisions
- ▶ Constraints
- ▶ Objective

Optimization seeks to choose some decisions to optimize (maximize or minimize) an objective subject to certain constraints.

## Common Framework

Given  $f, g_i, h_i : \mathbb{R}^n \mapsto \mathbb{R}$

$$Z = \underset{\mathbf{x}}{\text{minimize/maximize}} \quad f(\mathbf{x}) \quad (1a)$$

$$\text{subject to} \quad g_i(\mathbf{x}) \leq 0, \forall i = 1, 2, \dots, p \quad (1b)$$

$$g_j(\mathbf{x}) \geq 0, \forall j = 1, 2, \dots, q \quad (1c)$$

$$h_k(\mathbf{x}) = 0, \forall k = 1, 2, \dots, r \quad (1d)$$

- ▶ **Decisions:**  $\mathbf{x}$ , **Objective:**  $f(\mathbf{x})$ , and **Constraints:** (1b)-(1d)
- ▶ (1b), (1c), and (1d): set of " $\leq$ ", " $\geq$ ", and equality constraints
- ▶  $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : (1b) - (1d)\}$  define the **feasible region**.
- ▶ Any  $\hat{\mathbf{x}}$  satisfying all the constraints is a **feasible solution**.
- ▶ Any  $\mathbf{x}^* \in \mathcal{X}$  satisfying  $f(\mathbf{x}^*) \leq f(\mathbf{x}), \forall \mathbf{x} \in \mathcal{X}$  is an **optimal solution**.
- ▶  $f(\mathbf{x}^*)$  is known as **optimal objective value**.

## A few classes of optimization problems

- ▶ **Linear optimization:**  $f, g_i, h_i$  are all affine functions of continuous variables  $x$ .
- ▶ **Non-linear optimization:** At least one of  $f, g_i, h_i$  is non-linear function of continuous variables  $x$ .
  - **Convex optimization:** All functions are convex and feasible region is a convex set
- ▶ **(Mixed) Integer optimization:** Some of the variables  $x$  are restricted to be integers.
- ▶ **(Mixed) Integer Non-linear optimization:** Some of the variables  $x$  are restricted to be integers and at least one of  $f, g_i, h_i$  is non-linear.

Difficulty of solving above classes rises significantly as we go from above to below.



## A few definitions

**Definition** (Maximum) Let  $S \subseteq \mathbb{R}$ . We say that  $x$  is a maximum of  $S$  iff  $x \in S$  and  $x \geq y, \forall y \in S$ .

**Definition** (Minimum) Let  $S \subseteq \mathbb{R}$ . We say that  $x$  is a minimum of  $S$  iff  $x \in S$  and  $x \leq y, \forall y \in S$ .

**Definition** (Bounds) Let  $S \subseteq \mathbb{R}$ . We say that  $u$  is an upper bound of  $S$  iff  $u \geq x, \forall x \in S$ . Similarly,  $l$  is a lower bound of  $S$  iff  $l \leq x, \forall x \in S$ .

**Definition** (Supremum) Let  $S \subseteq \mathbb{R}$ . We define the supremum of  $S$  denoted by  $\mathbf{sup}(S)$  to be the smallest upper bound of  $S$ . If no such upper bound exists, then we set  $\mathbf{sup}(S) = +\infty$ .

**Definition** (Infimum) Let  $S \subseteq \mathbb{R}$ . We define the infimum of  $S$  denoted by  $\mathbf{inf}(S)$  to be the largest lower bound of  $S$ . If no such lower bound exists, then we set  $\mathbf{inf}(S) = -\infty$ .

**Definition** If  $x \in S$  such that  $x = \mathbf{sup}(S)$ , we say that supremum of  $S$  is **achieved** (which means that there is a maximum to the problem). Similar definition for whether infimum is achieved.

## General Formulation of LP

$$Z = \underset{\mathbf{x}}{\text{minimize/maximize}} \quad \mathbf{c}^T \mathbf{x} \quad (2a)$$

$$\text{subject to} \quad \mathbf{a}_i^T \mathbf{x} \leq b_i, \forall i \in C_1 \quad (2b)$$

$$\mathbf{a}_j^T \mathbf{x} \geq b_j, \forall j \in C_2 \quad (2c)$$

$$\mathbf{a}_k^T \mathbf{x} = b_k, \forall k \in C_3 \quad (2d)$$

$$x_u \geq 0, \forall u \in N_1 \quad (2e)$$

$$x_v \leq 0, \forall v \in N_2 \quad (2f)$$

$$x_w \text{ free}, \forall w \in N_3 \quad (2g)$$

where,  $C_1, C_2, C_3 \subseteq \{1, \dots, m\}$ ,  $N_1, N_2, N_3 \subseteq \{1, \dots, n\}$

## More definitions

**Definition** (Hyperplane)  $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T \mathbf{x} = b\}$

**Definition** (Halfspace)  $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T \mathbf{x} \geq b\}$

**Definition** (Polyhedron) A set  $P \subseteq \mathbb{R}^n$  is called a **polyhedron** if  $P$  is the intersection of a finite number of halfspaces.  $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$

**Definition** (Polytope) A bounded polyhedron is called a polytope.

**Question** Is  $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$  a polyhedron?

**Definition** (Convex Sets) A set  $S \subseteq \mathbb{R}^n$  is a convex set if for any  $\mathbf{x}, \mathbf{y} \in S$ , and  $\lambda \in [0, 1]$ , we have  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S$ .

**Question** Is polyhedron  $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$  a convex set?

**Definition** (Convex combination)  $\mathbf{x} \in \mathbb{R}^n$  is said to be convex combination of  $\mathbf{x}^1, \dots, \mathbf{x}^p \in \mathbb{R}^n$  if for  $\lambda_1, \dots, \lambda_p \geq 0$  s.t.  $\sum_i^n \lambda_i = 1$ ,  $\mathbf{x}$  can be expressed as  $\mathbf{x} = \sum_i^n \lambda_i \mathbf{x}^i$ .

**Definition** (Extreme point) Let  $P$  be a polyhedron. Then,  $\mathbf{x} \in P$  is an extreme point of  $P$  if we cannot express  $\mathbf{x}$  as a convex combination of other points in  $P$ .

## Theorem

Let  $P$  be a non-empty polyhedron. Consider LP  $\max\{\mathbf{c}^T \mathbf{x} \text{ s.t. } \mathbf{x} \in P\}$ . Suppose the LP has at least one optimal solution and  $P$  has at least one extreme point. Then, above LP has at least one extreme point of  $P$  that is an optimal solution.

## Possible states of optimization problems

An optimization problem may have the following states:

- ▶ Infeasible (max problems,  $Z = -\infty$  and min problems,  $Z = +\infty$ )
- ▶ Feasible, optimal value finite but not attainable
- ▶ Feasible, optimal value finite and attainable
- ▶ Feasible, but optimal value is unbounded (max problems,  $Z = +\infty$  and min problems,  $Z = -\infty$ )

# Outline

Motivation

Optimization Framework

**Reformulations**

Examples

## Pointwise maximum/minimum, no problem!

How to linearize functions such as  $\max_i \{\mathbf{a}_i^T \mathbf{x} + b_i\}$  &  $\min_i \{\mathbf{a}_i^T \mathbf{x} + b_i\}$ ?

► Define  $y = \max_i \{\mathbf{a}_i^T \mathbf{x} + b_i\} \implies y \geq \mathbf{a}_i^T \mathbf{x} + b_i, \forall i$

► Define  $y = \min_i \{\mathbf{a}_i^T \mathbf{x} + b_i\} \implies y \leq \mathbf{a}_i^T \mathbf{x} + b_i, \forall i$

How about the following problem?

$$Z = \underset{\mathbf{x} \geq 0}{\text{minimize}} \quad \|\mathbf{x}\|_1 = \sum_i |x_i| \quad (3a)$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad (3b)$$

► Note  $|x_i| = \max\{x_i, -x_i\}$ . Define  $y_i = |x_i|$

$$Z = \underset{\mathbf{x} \geq 0, \mathbf{y}}{\text{minimize}} \quad \sum_i y_i \quad (4a)$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad (4b)$$

$$y_i \geq x_i, \forall i \quad (4c)$$

$$y_i \geq -x_i, \forall i \quad (4d)$$

## Pointwise maximum/minimum, no problem!

How about the following problem?

$$Z = \underset{\mathbf{x} \geq 0}{\text{minimize}} \quad \|\mathbf{x}\|_\infty = \max_i \{|x_i|\} \quad (5a)$$

$$\text{subject to} \quad A\mathbf{x} = \mathbf{b} \quad (5b)$$

► Define  $y = \max_i \{|x_i|\}$

$$Z = \underset{\mathbf{x} \geq 0, y}{\text{minimize}} \quad y \quad (6a)$$

$$\text{subject to} \quad A\mathbf{x} = \mathbf{b} \quad (6b)$$

$$y \geq x_i, \forall i \quad (6c)$$

$$y \geq -x_i, \forall i \quad (6d)$$



# Linear Integer Program

$$Z = \underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T \mathbf{x} \quad (7a)$$

$$\text{subject to} \quad A\mathbf{x} = \mathbf{b} \quad (7b)$$

$$x_i \in \mathbb{Z}_+, i = 1, \dots, p \quad (7c)$$

$$x_i \in \mathbb{R}_+, i = p + 1, \dots, n \quad (7d)$$

- ▶ Generally, solving IP is more difficult than solving an LP. We use various tools from LP to approach this difficult problem.
- ▶ Better formulating the problem makes a lot of difference.

# Outline

Motivation

Optimization Framework

Reformulations

Examples

## Example

(Fleet sizing Problem) CEGE 5214 A transit agency is going to optimize its fleet size and type to maximize its revenue. Possible vehicle types are:

- ▶ Vans, capacity 6, purchase cost \$20, projected revenue \$96
- ▶ Regular buses, capacity 28, purchase cost \$120, projected revenue \$400
- ▶ Articulated buses, capacity 56, purchase cost \$220, projected revenue \$900

Constraints:

- ▶ Available budget is \$2,000
- ▶ The agency has 25 drivers who have 20% vacation/sick/no-show rate
- ▶ The fleet should provide a minimum capacity of 450
- ▶ At least 30% of the fleet should be vans for demand-responsive service
- ▶ At least 10 regular buses are needed for the fixed routes
- ▶ Exactly 2 articulated buses are needed for an express route

## Example

(Shortest path problem) Given a directed graph  $G(N, A)$ , cost of traversing links  $c : A \mapsto \mathbb{R}$ , find the shortest path from  $s \in N$  to  $t \in N$ .

## Example

(Maximum flow problem) Given a directed graph  $G(N, A)$ , cost of traversing links  $c : A \mapsto \mathbb{R}$ , and capacities of links  $u : A \mapsto \mathbb{R}$ , find the maximum flow possible to send from  $s \in N$  to  $t \in N$ .

## Example

(Transportation Problem) We have  $n$  factories each supplying  $a_i$  units of construction lumber and  $m$  cities each with  $b_i$  demand of lumber. If the transportation cost of each unit from factory  $i$  to city  $j$  is  $c_{ij}$ , formulate a program that minimizes the total transportation cost while serving the demand in all the cities.

Thank you!