Routing multiple flows through a network

Pramesh Kumar

IIT Delhi

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Suggested reading

- Goetschalckx Chapter 7

Outline

Introduction

- In a supply chain network, suppliers need to send products over transportation channels through various intermediate facilities to the customers.
- There may be capacity limits of transportation channels, intermediate facilities, and available goods.
- Such interaction between various parts of supply chain can be characterized using a graph $G(N, A)$, where N denotes various facilities and A represents the connection between different parts of the supply chain.
- The links may have capacity constraints.
- There may be cost of traversing the links.
- The overall supply of goods/services from sources to sinks is formulated as a network flow problem.

Outline

Maximum flow problem

Given a capacitated directed network $G(N, A)$ and capacity of links $u : A \mapsto \mathbb{R}$, find the maximum value of flow that can be sent between two special nodes, namely source $s \in N$ and sink $t \in N$, without exceeding the capacity of any link in the network.

Maximum flow problem example

What is the maximum number of bogeys manufactured in Detroit that can be shipped to a warehouse in San Francisco if there is a limit on how many compartments can be shipped across each link of the train network?

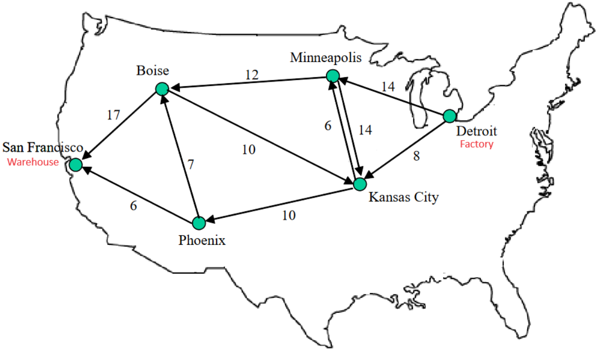

Figure: Golf cart shipping
(Source: https://ieda.ust.hk/dfaculty/ajay/courses/ieem101/lecs/graphs/graph-maxflow.pdf)

Flow

Definition (Flow). A flow in G is a real-valued function $x : A \mapsto \mathbb{R}$ that satisfies two properties:

1. Capacity constraints

$$0 \leq x_{ij} \leq u_{ij}, \forall (i,j) \in A \tag{1}$$

2. Flow conservation

$$\sum_{j \in FS(i)} x_{ij} - \sum_{j \in BS(i)} x_{ij} = 0, \quad \forall i \in N \backslash \{s,t\} \tag{2}$$

$$\sum_{j \in FS(i)} x_{sj} = v \tag{3}$$

$$\sum_{j \in BS(i)} x_{jt} = v \tag{4}$$

For the max flow problem, we need to maximize v.

Residual network corresponding to a flow

Residual capacity $r_{ij}(\mathbf{x}) = \begin{cases} u_{ij} - x_{ij} & \text{if } (i,j) \in A \\ x_{ji} & \text{if } (j,i) \in A \\ 0 & \text{otherwise} \end{cases}$

- Either one of the first two cases will occur (assumption 5).
- Residual network consists of links whose capacities represent how the flow can change on links.
- We only have links with positive residual capacities.
- For $(i,j) \in A$, even if do not have $(j,i) \in A$, we might still have (j,i) in the residual network. The purpose of creating this link to decrease the flow on $(i,j) \in A$ so as to increase the overall flow from s to t.

Definition (Augmenting path). A path with non-zero residual capacity is an augmenting path.

Example

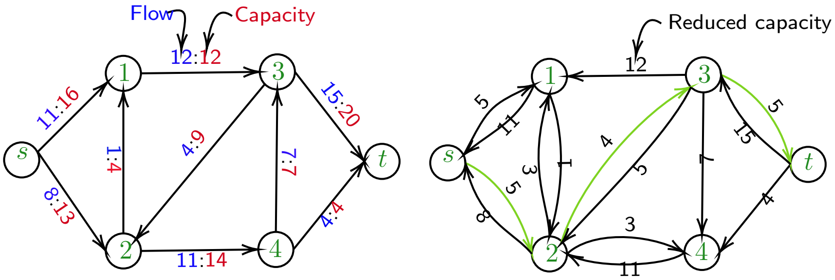

Figure: The first figure show the flow and capacities in the network. The second figure shows the residual network for the given flow. It further shows an augmenting path in green color.

Types of algorithms

1. **Augmenting path algorithms**
2. **Preflow-push algorithms**

Ford-Fulkerson method

procedure $\textsc{AugmentingPath}(G, c, u, s, t, x)$
  Initialize flow $\mathbf{x} = 0$
  while $\exists$ an augmenting path P in the residual network $G(\mathbf{x})$ **do**
    Augment the flow along P.
  end while
  return $\mathbf{x}$
end procedure

Example: Augmenting the flow

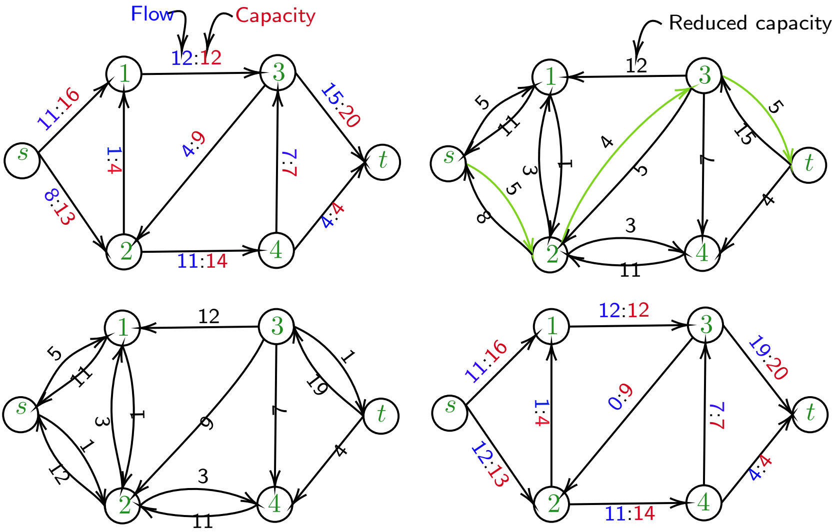

Figure: Augmenting the flow along path $s-2-3-t$

Cut

Definition ($s-t$ cut). An $s-t$ cut is a partition of nodes into two subsets S and $T = N\backslash S$ such that $s \in S$ and $t \in T$.

Definition (Flow across cut). The flow across an $s-t$ cut (S,T) is given as:

$$x(S,T) = \sum_{i\in S}\sum_{j\in T} x_{ij} - \sum_{i\in S}\sum_{j\in T} x_{ji} \tag{5}$$

Definition (Capacity of cut). The capacity of an $s-t$ cut (S,T) is given as:

$$u(S,T) = \sum_{i\in S}\sum_{j\in T} u_{ij} \tag{6}$$

Example

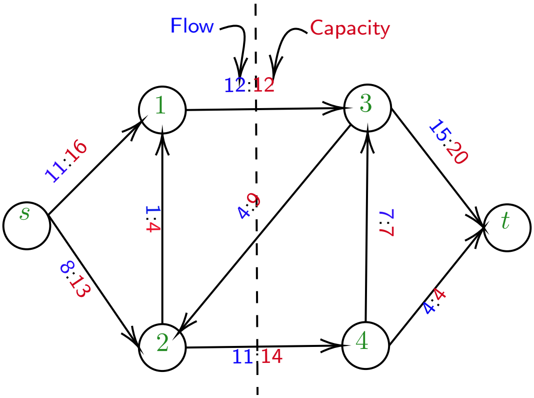

Figure: An $s-t$ cut

$S = \{s, 1, 2\}$ and $T = \{3, 4, t\}$.
$x(S,T) = x_{13} + x_{24} - x_{32} = 12 + 11 - 4 = 19$ and
$u(S,T) = u_{13} + u_{24} = 12 + 14 = 26$.

Minimum cut problem

Among all the $s-t$ cuts in the network, find one with minimum capacity.

LP formulation

Max flow problem

$$\max_{\mathbf{x},v} \quad v$$

$$\text{s.t.} \quad \sum_{j \in FS(i)} x_{ij} - \sum_{j \in BS(i)} x_{ji} = \begin{cases} v & \text{if } i = s \\ -v & \text{if } i = t \\ 0 & \forall i \in N \backslash \{s\} \end{cases}$$

$$0 \le x_{ij} \le u_{ij}, \forall (i,j) \in A$$

Min cut problem

$$\min_{\lambda,\mu} \sum_{(i,j) \in A} \lambda_{ij} u_{ij}$$

$$\text{s.t.} \quad \mu_i - \mu_j + \lambda_{ij} \ge 0, \forall (i,j) \in A$$

$$-\mu_s + \mu_t = 1$$

$$\lambda_{ij} \ge 0, \forall (i,j) \in A$$

Any cut (S,T) can be associated to the dual problem as $\mu_i = 0$, if $i \in S$ and $\mu_i = 1$, if $i \in T$. $\lambda_{ij} = 1$, if $i \in S, j \in T$, 0, otherwise

Max-flow min-cut theorem

Theorem (Strong duality)

If $\mathbf{x}$ be a flow in the network $G(N, A)$ with source s and sink t, then the following conditions are equivalent.

1. $v = \sum_{j \in FS(s)} x_{sj}$ *is a maximum flow in* G.
2. *The residual network* $G(\mathbf{x})$ *contains no augmenting path.*
3. $v = u(S, T)$ *for some cut* (S, T) *of* G.

Outline

Minimum cost flow problem

Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, lower and upper bounds (capacity) on the flow on links $l : A \mapsto \mathbb{R}$ and $u : A \mapsto \mathbb{R}$ resp., and supply/demand at each node $b : N \mapsto \mathbb{R}$, find the least cost shipment of a commodity.

LP formulation

Primal

$$\min_{\mathbf{x}} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j\in FS(i)} x_{ij} - \sum_{j\in BS(i)} x_{ji} = b(i), \forall i \in N$$

$$0 \leq x_{ij} \leq u_{ij}, \forall (i,j) \in A$$

where, if $b(i) > 0, b(i) < 0,$ and $b(i) = 0$, then i is called supply node, demand node, and transshipment node respectively.

Algorithms

- To save time, we are skipping algorithms for mincost flow problem. Those who are interested, please refer to this link.

Thank you!