Elementary definitions in graph theory

Pramesh Kumar

IIT Delhi

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Introduction

Definition (Network). A network is interconnection among set of items.

Internet network

Figure: Source: https://www.discovery.org/a/25/

Social network

Figure: Source: Medium

Highway network

Figure: Twin cities highway network

Transit network

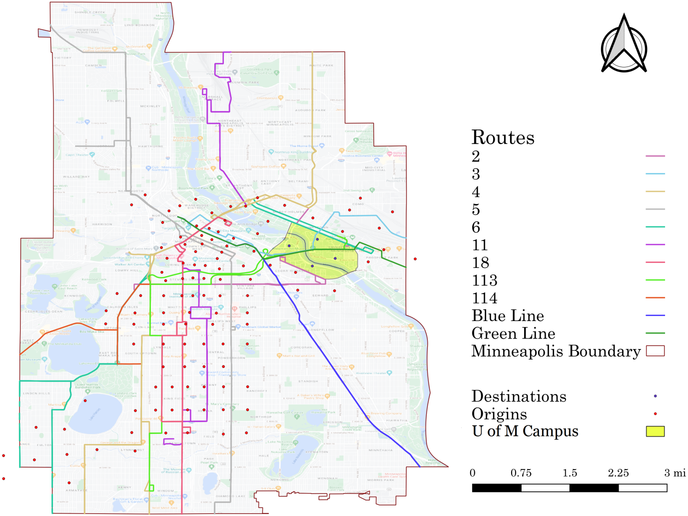


Figure: South Minneapolis transit network

Airline network

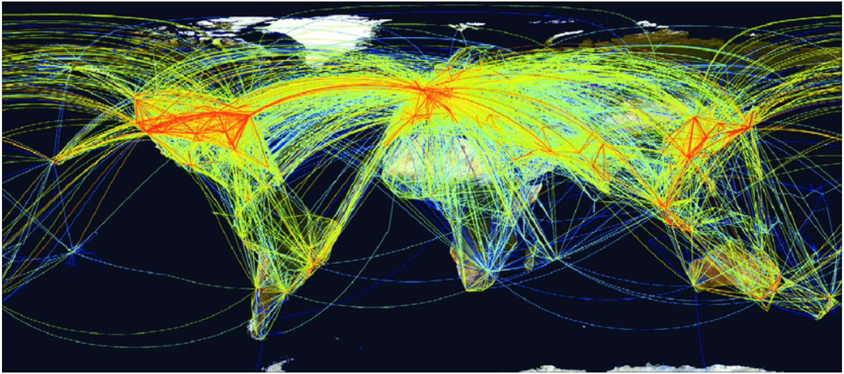
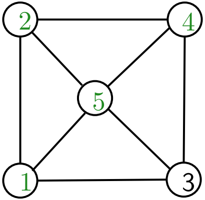


Figure: Source: Sarah Randolph on ResearchGate

Undirected graph

Definition (Undirected graph/network). An undirected graph G is a pair (N, A), where N is the set of nodes and A is the set of links whose elements are unordered pair of distinct nodes.

Example(s). $N = \{1, 2, 3, 4, 5\}$,
$A = \{(1, 2), (1, 3), (1, 5), (5, 4), (5, 3), (5, 2), (2, 4), (3, 4)\}$

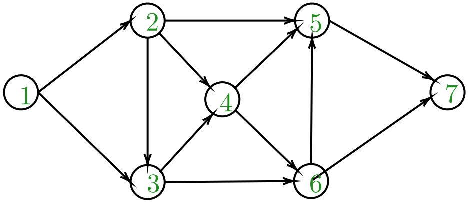


Remark. Let $|N| = n$. Then, $|E| = m \leq \frac{n(n-1)}{2}$.

Directed graph

Definition (Directed network/graph). A directed graph is pair (N, A), where N denotes the set of nodes/vertices and $A \subseteq N \times N$ denotes the set of links/edges/arcs whose elements are ordered pair of distinct nodes.

Example(s). $N = \{1, 2, 3, 4, 5, 6, 7\}$
$A = \{(1,2), (1,3), (2,3), (2,4), (2,5), (3,4), (3,6), (4,5), (4,6), (5,7), (6,5), (6,7)\}$



Definition (). If $e = (i, j) \in A$, then

1. i and j are endpoints of e.
2. i is the tail node and j is the head node of e.
3. (i, j) emanates from i and terminates at node j.
4. (i, j) is incident to nodes i and j.
5. (i, j) is outgoing link of node i and incoming link of node j.

Definition (Degree). The number of incoming and outgoing links of a node $i \in N$ are called indegree and outdegree respectively. The sum of indegree and outegree is called degree.

Definition (Multilinks). Two or more links with same head and tail nodes.

Definition (Loop). A link whose tail and head nodes are the same.

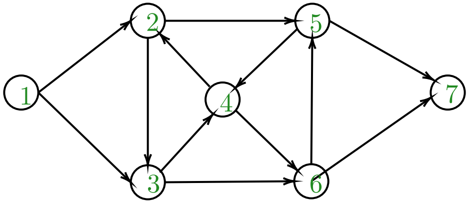
Note: In this course, we assume that graphs contain no loops or multiarcs.

Definition (Subgraph). A graph $G^{'}(N^{'}, A^{'})$ is a subgraph of $G(N, A)$ if $N^{'} \subseteq N$ and $A^{'} \subseteq A$. A subgraph $G^{'}(N^{'}, A^{'})$ of $G(N, A)$ is said to be induced by $N^{'}$ if $A^{'}$ contains links with their end points in $N^{'}$.

Definition (Walk). A collection of links $W = \{(i_1, j_1), \cdots, (i_q, j_q)\}$ is an $s - t$ walk if

1. $u_1 = s$
2. $v_i = u_{i+1}, \forall i = 1, ..., q - 1$
3. $u_q = t$

Example(s).



$W_1 = \{(1,2),(2,5),(5,7)\}$,
$W_2 = \{(1,2),(2,3),(3,4),(4,2),(2,5),(5,7)\}$,
$W_3 = \{(1,3),(3,6),(6,5),(5,4),(4,6),(6,7)\}$
are all exmples of $1-7$ walks.

Definition (Path). An $s-t$ path is an $s-t$ walk without any repeated nodes.
In above example, W_1 is a $1-7$ path while W_2 and W_3 are not.

Definition (Cycle). A cycle is path where its first and last nodes are same.

Definition (Tour). A tour is a cycle including all nodes of the graph.

Definition (Acyclic graph). A graph without any cycles is acyclic.

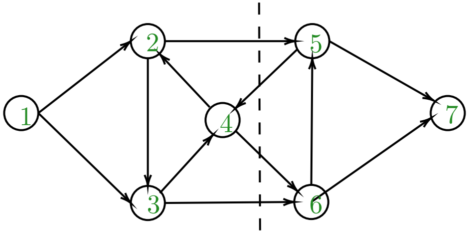
Definition ().

1. Nodes $i \in N$ and $j \in N$ are said to be connected if there exists at least one path between i and j
2. A graph is said to be connected graph if every pair of its nodes are connected. Otherwise, the graph is called disconnected.

Definition (Cut). A cut is a partition of nodes into two subsets S and $\bar{S} = N \backslash S$.

- Each cut defines a set of links with one endpoint in S and another in $\bar{S}$. This set of links is denoted by $(S, \bar{S})$.
- An $s - t$ cut is a cut $(S, \bar{S})$ with $s \in S$ and $t \in \bar{S}$.

Example(s).



$S = \{1, 2, 3, 4\}, \bar{S} = \{5, 6, 7\}$, and $(S, \bar{S}) = \{(2, 5), (5, 4), (4, 6), (3, 6)\}$ defines a $1 - 7$ cut.

Definition (Tree). A tree is a connected graph that contains no cycles.

Proposition

1. *A tree on n nodes contains exactly $n - 1$ links.*
2. *A tree has at least 2 leaf nodes (i.e., nodes with degree 1).*
3. *Every pair of nodes are connected by a unique path.*

Proof.

1. (Proof by induction) Let $P(n)$ be the statement that a tree on n nodes contains exactly $n - 1$ links. $P(1) = 0$ since there is only one node and a link requires at least two nodes. Let us assume that $P(k)$ is true, i.e., a tree on k nodes contains exactly $k - 1$ links. Then, we can add another node to this graph with one link and that would still be a tree with k links, which means that $P(k + 1)$ is true.
2. Assuming $n < \infty$, we prove this by contradiction. Assume that a tree on n nodes has only one leaf u. Then, find the longest path from u in the tree. The longest path cannot end at u because that is not a path but cycle. Let us assume that it ends at v. If v has degree 1 then we are done. If it has degree 2, then it is not a longest path.
3. Proof by induction. Not possible to add another node without creating a cycle.

□

Bipartite graphs

Definition (Bipartite graph). A graph $G(N, A)$ is bipartite if we can partition N into two subsets N_1 and N_2 such that for every link $(i, j) \in A$, we have either $i \in N_1$ and $j \in N_2$ or $j \in N_1$ and $i \in N_2$.

Proposition

A graph G is bipartite if and only if every cycle in G contains an even number of links.

Proof.

($\Rightarrow$) Assume that G is bipartite. Then, every step of a walk will take you either from N_1 to N_2 or N_2 to N_1. To form a cycle, you need to come back where you started requires even number of steps.
($\Leftarrow$) Assume that every cycle in G is even. Then, starting from one node $u \in C_1$ along a path/cycle, put nodes at odd distance in C_2 and nodes at even distance in C_1. Do this for every connected components of G. We cannot have link between two nodes within C_1 or C_2, otherwise cycle will be odd. $\square$

Network representation

- The performance of a network algorithm depends not only on the algorithm but also on which data structure we use to store the network.
- We need to store how nodes are connected as well as capacities or costs associated to links.

Data structures

1. Node-link incidence matrix
2. Node-node adjacency matrix
3. Adjacency list
4. Forward (Backward) Star