Vehicle routing problem

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Suggested reading

- Goetschalckx Chapter 9
- Larson, R.C. and Odoni, A.R., 1981. Urban operations research Chapter 6

Multi-route problems

- In many practical situations, we are given a task of routing multiple vehicles to perform a shared activity.
- Examples:
  - City district performing waste collection across various sub-districts using a fleet of vehicles.
  - Snow plowing on various highways using a fleet of vehicles.
  - Routing of multiple vehicles to drop-off packages at various residential locations in the city.
  - Pick up and deliver passengers from between various origin-destination pairs using a fleet of vehicles.
- Multi-route problems are generally harder to solve than the single-tour problems.
- Following strategies have been used.
  - Partition the region into sub-regions and then design optimal single vehicle route within each sub-region. This strategy is also called cluster first, route second. This strategy is more popular than the one below.
  - Design a single grand optimal route for the whole region and then subdivide it into number of subroutes each to be served by separate vehicle. This strategy is also called route first, cluster second.

$m-$TSP

$m-$TSP can be transformed into one-salesman TSP using the following trick:

- Let $O, 1, \cdots, n$ be the cities to be visited
- Replace O with m copies, say $O_1, \cdots O_m$ by keeping
  - $d(O_i, x) = d(O, x), \forall x = 1, \cdots, n, \forall i = 1, \cdots, m$ and
  - $d(O_i, O_j) = \infty, \forall i = 1, \cdots, m, \forall j = 1, \cdots, m$
- Solve one-salesman TSP on the modified graph
- Merge $O_1, \cdots O_m$ to O to get the optimal solution to $m-$TSP.

Vehicle routing

- One of challenging tasks in transportation logistics is to manage a fleet of vehicles to provide transportation services.
- The transportation services can be used for pickup or drop off (or both) of passengers/goods from/to number of locations.
- The transportation operations can be subjected to several constraints:
  - Vehicle capacity (weight, volume or both)
  - Limit on delivery/pickup times (time windows, travel time)
  - Precedence constraints (e.g., all goods need to be picked up before they can be dropped off)
  - Service constraints
- At operations level, generally, two types of decisions have to be made
  - Assignment of transportation service requests to vehicles
  - Evaluating the sequence of stops to be visited by each vehicle
- Tactical decisions include the size and composition of vehicles to acquire.

Vehicle Routing Variants

- Capacitated Vehicle Routing Problems (CVRP): A number of customers with known locations and demand have to be serviced by a fleet of vehicles of identical capacity and that all vehicles will start and end their trip at a single depot. The objective is to minimize the total cost of routes performed by the vehicles.
- Vehicle Routing Problem with Backhauling (VRPB): Two types of locations need to be visited. One set is the set of pickups and other set is the set of drop-offs. The drop-offs need to be performed before pickups.
- Mixed Pickup and Delivery Problem The deliveries and pickups can be intermixed. For example, Coke bottles deliveries and pickups.
- Vehicle Routing Problem with Time Windows (VRPTW) It includes time window during which each request need to be served. The time window can be hard or soft constraints (dealt with penalty).
- Inventory Vehicle Routing Problem (IVRP) The objective it to minimize the sum of transportation as well as holding inventory costs while avoiding stock-outs and storage capacity limits.

Capacitated vehicle routing problem (CVRP)

- A set of homogeneous customers $1, \cdots, n$ requesting delivery of a product. Their demand is unsplittable.
- A company has K identical vehicles (initially located at a depot) each having capacity C.
- Let $G(N, A)$ be an directed graph, where $N = \{0, 1, \cdots, n\}$ denotes the set of nodes. Node 0 is the depot location and nodes $1, \cdots, n$ are the customer locations.
- Let c_{ij} be the cost of traveling from i to j. It is possible that $c_{ij} \neq c_{ji}$.
- Let q_i be the demand of customer i

Definition (CVRP). Determine the tours of the K vehicles such that

1. Each tour departs from depot (node 0).
2. Each customer is served by exactly one vehicle.
3. Capacity of each vehicle is not exceeded.
4. Demand at each customer location is satisfied.

and the total cost of serving customer requests is minimized.

Two-index formulation

Let $x_{ij} = 1,$ if $(i,j) \in A$ belongs to the tour$, 0,$ otherwise

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{j\in FS(i)} x_{ij} = 1, \forall i \in N\backslash\{0\} \\
& \sum_{j\in BS(i)} x_{ji} = 1, \forall i \in N\backslash\{0\} \\
& \sum_{j\in BS(0)} x_{j0} = K \\
& \sum_{j\in FS(0)} x_{0j} = K \\
& \sum_{i\in S, j\notin S} x_{ij} \geq r(S)^{*}, \forall S \subseteq N\backslash\{0\} : S \neq \phi \\
& x_{ij} \in \{0,1\}, \forall i \in N, \forall (i,j) \in A
\end{aligned}
$$

* $r(S) = \lceil \frac{\sum_i q_i}{C} \rceil$ is the minimum number of vehicle routes required to serve S.

CVRP MTZ three-index formulation

- Two-index formulation cannot model vehicle specific information such as capacities, associated depots, and costs.
- In three-index formation, depot 0 is replaced by two nodes o and d (start and end of a route).
- Decision variables are as follows:
- $x_{ijk} = \begin{cases} 1, & \text{if vehicle } k \text{ moves over the arc } (i,j) \\ 0, & \text{otherwise} \end{cases}$
- $u_{ik} =$ accumulated demand already distributed by vehicle k when arriving at node i
- $y_{ik} = \begin{cases} 1, & \text{if vehicle } k \text{ visits node } i \\ 0, & \text{otherwise} \end{cases}$

CVRP MTZ three-index formulation

$$\underset{\mathbf{x,\,y,\,u}}{\text{minimize}} \quad \sum_{k=1}^{K} \sum_{(i,j)\in A} c_{ijk} x_{ijk}$$

$$\text{s.t.} \quad \sum_{k=1}^{K} y_{ik} = 1, \forall i \in N$$

$$\sum_{j \in FS(i)} x_{ijk} - \sum_{j \in BS(i)} x_{jik} = \begin{cases} 1, & \text{if } i = o, \\ 0, & \text{if } i \in N, \end{cases}, \forall i \in N \cup \{o\}, \forall k = 1,$$

$$\sum_{j \in FS(i)} x_{ijk} = y_{ik}, \forall i \in N \cup \{o\}, \forall k = 1, \cdots, K$$

$$\sum_{j \in BS(d)} x_{jdk} = y_{dk}, \forall k = 1, \cdots, K$$

$$u_{ik} - u_{jk} + q_j \leq Q(1 - x_{ijk}), \forall (i,j) \in A, \forall k = 1, \cdots, K$$

$$q_i \leq u_{ik} \leq Q, \forall i \in N \cup \{o, d\}, \forall k = 1, \cdots, K$$

$$y_{ik} \in \{0, 1\}, \forall i \in N \cup \{o, d\}, \forall k = 1, \cdots, K$$

$$x_{ijk} \in \{0, 1\}, \forall (i,j) \in A, \forall k = 1, \cdots, K$$

Clarke-Wright heuristic

- Suppose that initial solution to VRP consists of using $|N|$ vehicles and dispatching one vehicle to each one of the $|N|$ demand points from the depot 0.
- However, if we use a single vehicle to serve two points, say i and j, on a single trip, the total cost is reduced by the amount

$$
\begin{aligned}
s(i,j) &= 2d(0,i) + 2d(0,j) - [d(0,i) + d(i,j) + d(0,j)] &(1)\\
&= d(0,i) + d(0,j) - d(i,j) &(2)
\end{aligned}
$$

- $s(i,j)$ is known as *savings* resulting from combining points i and j into a single tour. The larger the $s(i,j)$, the more beneficial is to combine i and j in a single tour. However, they cannot be combined if one or more constraints are violated.

Clarke-Wright savings algorithm

- Calculate the savings
  $s(i,j) = d(0,i) + d(0,j) - d(i,j), \forall (i,j) \in N \times N$.
- Rank the savings $s(i,j)$ in decreasing order (call it *savings list*). Process the pairs (i,j) one-by-one. For a given (i,j), make sure no constraints are violated. Then, one the following three cases will occur:
  1. Neither i nor j have been assigned to a route, in which case, a new route is initiated including both i and j
  2. Exactly one of the following two point have been included in an existing route and that point is not interior to that route (a point is interior to a route if it not adjacent to 0 in the sequence of the route), in which case the link (i,j) is added to the same route.
  3. Both i and j have already been included in two different existing routes and neither point is interior to its route, in which two routes are merged.
- Continue until all the pairs (i,j) have been processed.

Cluster first and route second

- $\mathfrak{c}_k$: cost of using vehicle k
- $z_{ik} = 1,$ if customer i is assigned to vehicle k, 0, otherwise
- $z_{0k} = 1,$ if vehicle k is used. 0, otherwise

$$
\begin{aligned}
\underset{\mathbf{x},\,\mathbf{y},\,\mathbf{u}}{\text{minimize}} \quad & \sum_{k=1}^{K} \mathfrak{c}_k z_{0k} \\
\text{s.t.} \quad & \sum_{i \in N} q_i z_{ik} \le Q z_{0k}, \forall k = 1, \cdots, K \\
& \sum_{k=1}^{K} z_{ik} = 1, \forall i \in N \\
& z_{ik} \in \{0, 1\}, \forall i \in N \cup \{0\}, \forall k = 1, \cdots, K
\end{aligned}
$$

- After the assignment, one can use exact/heuristics designed for TSP for routing individual vehicles.

Final thoughts

- There is so much to study about VRP.
- This is just an introduction.
- Following is a good resource to further study this problem: Toth, Paolo, and Daniele Vigo, eds. Vehicle routing: problems, methods, and applications. Society for industrial and applied mathematics, 2014.

Thank you!