Modeling Linear Optimization Problems

Pramesh Kumar

IIT Delhi

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Outline

What is optimization?

(Merriam-Webster Dictionary) An act, process, or methodology of making something (such as a design, system, or decision) as fully perfect, functional, or effective as possible.

Example 1

(Maximum Area Problem)
You have 80 meters of wire and want to enclose a rectangle as large as possible (in area). How should you do it?

Example 2

(Production Problem) A factory can produce two products, A and B. The production of each item of A takes 2 hours, and that of item B takes 7 hours. Further, each item of products A and B takes 22 and 41 ft^3 storage capacity, respectively. The manager gets a profit of \$30 and \$50 by producing each item of A and B resp. Assuming that there is an 88-hour limit on the number of hours of operating the factory and the maximum storage capacity of the factory is 9,000ft^3, how many items of A and B should the manager decide to produce to maximize the profit?

Outline

Common Framework

Components of an optimization problem

- Decisions
- Constraints
- Objective

Optimization seeks to choose some decisions to optimize (maximize or minimize) an objective subject to certain constraints.

Common Framework

Given $f, g_i, h_i : \mathbb{R}^n \mapsto \mathbb{R}$

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize/maximize}} \quad & f(\mathbf{x}) && \text{(1a)}\\
\text{subject to} \quad & g_i(\mathbf{x}) \leq 0, \forall i = 1, 2, ..., p && \text{(1b)}\\
& g_j(\mathbf{x}) \geq 0, \forall j = 1, 2, ..., q && \text{(1c)}\\
& h_k(\mathbf{x}) = 0, \forall k = 1, 2, ..., r && \text{(1d)}
\end{aligned}
$$

- Decisions: $\mathbf{x}$, Objective: $f(\mathbf{x})$, and Constraints: (1b)-(1d)
- (1b), (1c), and (1d): set of "$\leq$", "$\geq$", and equality constraints
- $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : (1b) - (1d)\}$ define the feasible region.
- Any $\hat{\mathbf{x}}$ satisfying all the constraints is a feasible solution.
- Any $\mathbf{x}^* \in \mathcal{X}$ satisfying $f(\mathbf{x}^*) \leq f(\mathbf{x}), \forall \mathbf{x} \in \mathcal{X}$ is an optimal solution.
- $f(\mathbf{x}^*)$ is known as optimal objective value.

A few classes of optimization problems

- Linear optimization: f, g_i, h_i are all affine functions of continuous variables x.
- Non-linear optimization: At least one of f, g_i, h_i is non-linear function of continuous variables x.
  - Convex optimization: All functions are convex and feasible region is a convex set
- (Mixed) Integer optimization: Some of the variables x are restricted to be integers.
- (Mixed) Integer Non-linear optimization: Some of the variables x are restricted to be integers and at least one of f, g_i, h_i is non-linear.

Difficulty of solving above classes rises significantly as we go from above to below.

A few definitions

Definition (Maximum) Let $S \subseteq \mathbb{R}$. We say that x is a maximum of S iff $x \in S$ and $x \geq y, \forall y \in S$.

Definition (Minimum) Let $S \subseteq \mathbb{R}$. We say that x is a minimum of S iff $x \in S$ and $x \leq y, \forall y \in S$.

Definition (Bounds) Let $S \subseteq \mathbb{R}$. We say that u is an upper bound of S iff $u \geq x, \forall x \in S$. Similarly, l is a lower bound of S iff $l \leq x, \forall x \in S$.

General Formulation of LP

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize/maximize}} \quad & \mathbf{c}^T\mathbf{x} && \text{(2a)}\\
\text{subject to} \quad & \mathbf{a}_i^T\mathbf{x} \leq b_i, \forall i \in C_1 && \text{(2b)}\\
& \mathbf{a}_j^T\mathbf{x} \geq b_j, \forall j \in C_2 && \text{(2c)}\\
& \mathbf{a}_k^T\mathbf{x} = b_k, \forall k \in C_3 && \text{(2d)}\\
& x_u \geq 0, \forall u \in N_1 && \text{(2e)}\\
& x_v \leq 0, \forall v \in N_2 && \text{(2f)}\\
& x_w \text{ free }, \forall w \in N_3 && \text{(2g)}
\end{aligned}
$$

where, $C_1, C_2, C_3 \subseteq \{1, ..., m\}$, $N_1, N_2, N_3 \subseteq \{1, ..., n\}$

More definitions

Definition (Hyperplane) $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$

Definition (Halfspace) $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \geq b\}$

Definition (Polyhedron) A set $P \subseteq \mathbb{R}^n$ is called a polyhedron if P is the intersection of a finite number of halfspaces. $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$

Definition (Polytope) A bounded polyhedron is called a polytope.
Question Is $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ a polyhedron?

Definition (Convex Sets) A set $S \subseteq \mathbb{R}^n$ is a convex set if for any $\mathbf{x}, \mathbf{y} \in S$, and $\lambda \in [0, 1]$, we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$.
Question Is polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ a convex set?

Definition (Convex combination) $\mathbf{x} \in \mathbb{R}^n$ is said to be convex combination of $\mathbf{x}^1, ..., \mathbf{x}^p \in \mathbb{R}^n$ if for $\lambda_1, ..., \lambda_p \geq 0$ s.t. $\sum_i^n \lambda_i = 1$, $\mathbf{x}$ can be expressed as $\mathbf{x} = \sum_i^n \lambda_i \mathbf{x}^i$.

Definition (Extreme point) Let P be a polyhedron. Then, $\mathbf{x} \in P$ is an extreme point of P if we cannot express $\mathbf{x}$ as a convex combination of other points in P.

Theorem

Let P be a non-empty polyhedron. Consider LP $\max\{\mathbf{c}^T\mathbf{x} \textit{ s.t. } \mathbf{x} \in P\}$. Suppose the LP has at least one optimal solution and P has at least one extreme point. Then, above LP has at least one extreme point of P that is an optimal solution.

Possible states of optimization problems

An optimization problem may have the following states:

- Infeasible (max problems, $Z = -\infty$ and min problems, $Z = +\infty$)
- Feasible, optimal value finite but not attainable
- Feasible, optimal value finite and attainable
- Feasible, but optimal value is unbounded (max problems, $Z = +\infty$ and min problems, $Z = -\infty$)

Outline

Maximum flow problem

(Maximum flow problem) Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, and capacities of links $u : A \mapsto \mathbb{R}$, find the maximum flow possible to send from $s \in N$ to $t \in N$.

Minimum cost flow problem

(Minimum cost flow problem) Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, lower and upper bounds (capacity) on the flow on links $l : A \mapsto \mathbb{R}$ and $u : A \mapsto \mathbb{R}$ resp., and supply/demand at each node $b : N \mapsto \mathbb{R}$, find the least cost shipment of a commodity. Note $b(i) > 0$ for a supply nodes, $b(i) < 0$ for demand nodes, and $b(i) = 0$ for transshipment nodes.

Integer Problems

Linear Integer Program

For $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $\mathbf{b} \in \mathbb{R}^m$

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T \mathbf{x} && \text{(3a)} \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} && \text{(3b)} \\
& x_i \in \mathbb{Z}_+, i = 1, ..., p && \text{(3c)} \\
& x_i \in \mathbb{R}_+, i = p+1, ..., n && \text{(3d)}
\end{aligned}
$$

- p decision variables are integers
- $n - p$ decision variables are continuous

Binary Integer Program

$$Z = \underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{4a}$$

$$\text{subject to} \quad A\mathbf{x} = \mathbf{b} \tag{4b}$$

$$x_i \in \{0, 1\}, \forall i = 1, ..., n \tag{4c}$$

- All decision variables can either take value 1 or 0.

Example: Knapsack Problem

Given a set of items N, each with a weight w_i and a value a_i, determine which items to include in the collection so that the total weight is less than or equal to a given limit W and the total value is as large as possible.

Let $x_i = \begin{cases} 1 & \text{if } i \text{ is picked} \\ 0 & \text{otherwise} \end{cases}$

$$Z = \underset{\mathbf{x}}{\text{maximize}} \quad \sum_{i=1}^{n} a_i x_i \tag{5a}$$

$$\text{subject to} \quad \sum_{i=1}^{n} w_i x_i \leq W \tag{5b}$$

$$x_i \in \{0, 1\}, \forall i = 1, ..., n \tag{5c}$$

Example: Uncapacitated Facility Location

Given a set of potential depots $N = \{1, \ldots, n\}$ and a set $M = \{1, \ldots, m\}$ of clients, suppose there is a fixed cost f_j associated with the opening of depot j, and a transportation cost c_{ij} if all of client i's order is delivered from depot j. The problem is to decide which depots to open and which depot serves each client so as to minimize the sum of the fixed and transportation costs.

Let $y_j = \begin{cases} 1 & \text{if depot } j \text{ is opened} \\ 0 & \text{otherwise} \end{cases}$ $\quad x_{ij} = \begin{cases} 1 & \text{if } i\text{'s order are served from depot } j \\ 0 & \text{otherwise} \end{cases}$

$$\underset{\mathbf{x},\mathbf{y}}{\text{minimize}} \quad \sum_{j \in N} \sum_{i \in N} c_{ij} x_{ij} + \sum_{j \in n} f_j y_j \tag{6}$$

$$\text{subject to} \quad \sum_{j \in N} x_{ij} = 1, \forall i \in M \tag{7}$$

$$\sum_{i \in M} x_{ij} \leq m y_j, \forall j \in N \tag{8}$$

$$x_{ij} \geq 0, \forall i \in M, \forall j \in N \tag{9}$$

$$y_j \in \{0, 1\}, \forall j \in N \tag{10}$$

where, m is a large positive integer.

First attempt to solving an integer program

Let's relax the integral constraints and solve the linear program (which is easy) and then find an integral solution closer to the optimal solution (e.g., by rounding off) to the linear program.

- Such LP is called the LP relaxation of the integer program.
- It may work in some cases.
- In other cases, it may not even find a feasible solution, forget about the optimal.

Example

Integer Program

$$\begin{aligned} \underset{x_1,x_2}{\text{maximize}} \quad & 9x_1 + 17x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \le 11 \\ & 3x_2 \le 11 \\ & \boxed{x_1, x_2 \in \mathbb{Z}_+} \end{aligned}$$

LP relaxation

$$\begin{aligned} \underset{x_1,x_2}{\text{maximize}} \quad & 9x_1 + 17x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \le 11 \\ & 3x_2 \le 11 \\ & \boxed{x_1 \ge 0, x_2 \ge 0} \end{aligned}$$

Example

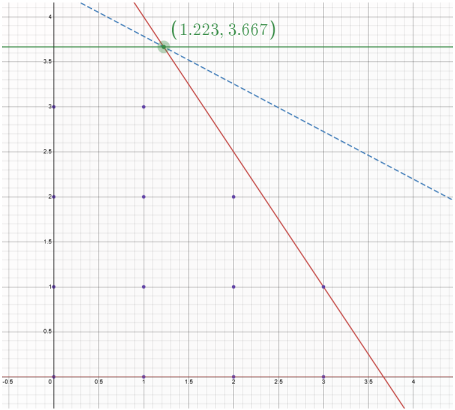

Figure: The feasible region is shown using points and the objective function is shown using dashed line

- The optimal solution of LP relaxation is $(1.223, 3.667)$ with objective value $=$ 73.346.
- If we try to round it, we get $(1, 4)$ which is not even a feasible solution.

Good news

Theorem

If the optimal solution to the LP relaxation is feasible to IP, then it must be optimal to the IP.

Question. When does solving the LP relaxation gives integral solution?

Totally unimodularity

Definition (Totally unimodular). A matrix A is totally unimodular (TU) if every square sub-matrix of A has determinant 1, -1, or 0.

Theorem (Sufficient condition for TU)

Let $A \in \mathbb{Z}^{m \times n}$. Then, A is TU if

1. *Each entry in A, i.e., a_{ij} is 0, 1, or -1.*
2. *Each column of A has at most two non-zero entries.*
3. *There exists a partition (M_1, M_2) of the set of rows $M = \{1, 2, \cdots, m\}$ such that each column j containing two non-zero entries satisfies $\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0$.*

Example(s). The incidence matrix of a directed graph is TU.

Example

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Let $M_1 = \{1, 2, 3\}$ and $M_2 = \{4\}$. Then

$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0, \forall j$$

.

Therefore, A is TU.

Theorem (Hoffman and Kruskal's Theorem)

Let $A \in \mathbb{Z}^{m \times n}$*. The polyhedron* $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$ *is integral for every* $b \in \mathbb{Z}^m$ *if and only if* A *is totally unimodular.*

Remark. The common misconception is that the only way you can get integral polyhedra if A is TU; not true.

Solving IP

- Generally, solving IP is hard.
- There is no polynomial time algorithm to solve an IP.
- Modern solvers such as gurobi use branch-and-cut with other techniques (preprocessing, etc.) to solve these.

Thank you!